SUPPLEMENT I

THE LOGIC OF RELATIVITY

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In order to be able to deal with such quantities as are involved in the measurement of motion, time, velocity, etc., or indeed in the quantitative analysis of any physical phenomena, it is necessary to have some system or systems of reference with respect to which measurements can be made. Let us consider any set of things consisting of objects and any kind of physical quantities whatever, as electric charges or magnets or light-sources or telescopes or other objects and instruments, each of which is at rest with respect to each of the others. Let us suppose that among the objects are clocks, to be used for measuring time, and rods or rules to be used for measuring length, and that time and length may be measured at any desired instant and any assigned place. Such a set of objects and quantities and instruments, including the equipment for measuring time and length, all being at rest relatively to each other, we shall call a system of reference. Such a system we shall denote by S. In case we have to deal at once with two or more systems of reference we shall denote them by S, S', S_1, S_2, . . .

In this definition of systems of reference nothing specific has been said about the units of length and of time. If we were dealing with our usual principles of mechanics we might pass over such a matter without any feeling of difficulty about it; it would be sufficient to proceed in accordance with our intuitive conceptions of time and length. But in the theory of relativity these appear in a new light. We can not proceed with confident dependence upon our intuition. On the other hand we shall not attempt to give explicit definitions of units of time and length. We shall proceed from certain principles or postulates, presently to be stated, to an analysis of time and length and so arrive at a suitable precision of these conceptions by means of certain guiding principles. It will be seen that it is not far from the truth to say that our fundamental terms are defined implicitly and indirectly by means of the statements made about them and accepted initially as valid and that they may mean anything which is consistent with the truth of these fundamental principles and postulates.

The restricted principle of relativity may now be stated in the following form:

RESTRICTED PRINCIPLE OF RELATIVITY. *If S_1 and S_2 are two systems of reference having with respect to each other a uniform unaccelerated motion, then natural phenomena run their course with respect to S_2 in accordance with precisely the same general laws as with respect to S_1.*

This principle says nothing about the suitability of any particular system of reference for the convenient expression of the laws of nature; but it does say that if either S_1 or S_2 is suitable the other is equally suitable, the relative motion of the two being unaccelerated.

In order to bring into suitable relations the measurements made on one system of reference and those made on another it is necessary to have some agreement as to the correspondence of units on the two systems. Accordingly we shall make the following assumption concerning the correspondence of units:

PRINCIPLE OF CORRESPONDENCE OF UNITS. *The units of any two systems S_1 and S_2 are such that the same numerical result will be obtained in measuring with the units of S_1 a quantity L_1 and with the units of S_2 a quantity L_2 when the relation of L_1 to S_1 is precisely the same as that of L_2 to S_2.*

We shall agree that the restricted principle of relativity is to be understood in a sense which implies this assumption concerning the correspondence of units; that is, the latter will be taken as a more precise formulation of a part of the content of the former. It is clear that the possibility of realizing this latter is taken for granted in the Galileo-Newtonian mechanics; it is often passed over without remark although it is a profound fact and is a part of the essential basis of any theory of motion.

It is a grave question whether the restricted principle of relativity can be maintained in the interpretation of natural phenomena. Indeed in the more general theory of relativity, to be taken up later, it is treated merely as a sort of approximation to a more comprehensive principle—an approximation strictly valid only in the absence of a gravitational field but very close to the truth for a wide variety of phenomena including most of those which are purely terrestrial.

There are two particular characteristic postulates, or ‘laws of nature’, lying at the base of the restricted theory of relativity. These may be stated as follows;

POSTULATE M. *The unaccelerated motion of a system of reference S can not be detected by observations made on S alone, the units of measurement being those belonging to S.*

POSTULATE R. *The velocity of light, in free space, measured on an unaccelerated system of reference S by means of units belonging to S, is independent of the velocity of S and of the unaccelerated velocity of the light-source.*

For these two particular postulates there is the strongest possible experimental evidence. Everything known points toward their truth, and there is nothing known which in any way seems to be in disagreement with them. It is to be observed that they apply only to the ideal case, that is, the case in which there is supposed to be no gravitational field.

For the development of the restricted theory of relativity there are three additional necessary postulates, or ‘laws of nature;’ those that theory shares in common with the Galileo-Newtonian mechanics. Such assumptions in some form are essential to the initial arguments and to the conclusions which are drawn by means of them. To the present writer it seems to be preferable to have these assumptions explicitly stated. They may be put into the following form:

POSTULATE V. *If the velocity of a system of reference S_2 relative to a system of reference S_1 is measured by means of the units belonging to S_1 and if the velocity of S_1 relative to S_2 is measured by means of the units belonging to S_2 the two results will agree in numerical value.*

POSTULATE T. *If two systems of reference S_1 and S_2 move with unaccelerated relative velocity and if a body moves relatively to one of the systems in a straight line with unaccelerated velocity then it also moves in a straight line relatively to the other and with unaccelerated velocity.*

POSTULATE L. *If two systems of reference S_1 and S_2 move with unaccelerated relative velocity and if a line segment l is perpendicular to the line of relative motion of S_1 and S_2 and is fixed to one of these systems, then the length of I measured by means of the units belonging to S_1 will be the same as its length measured by means of the units belonging to S_2.*

We now have before us the logical basis upon which may be built the restricted theory of relativity in all its details. It has been put in essentially the same form as that employed in my "Theory of Relativity" (published by Wiley and Sons, New York) and in my earlier articles in "The Physical Review". Reference may be made to the book named for the detailed development of the theory. Here we shall attempt to sketch only the progress of ideas and to indicate the main conclusions.

The first thing to be done in developing the theory on this basis is to consider carefully the relation between the time units of the two systems. The following remarkable conclusion is reached by a process of reasoning which is fully cogent in character:

If two systems of reference S_1 and S_2 move with a relative velocity v and β is the ratio v/c of v to the velocity c of light as measured on either system, then to an observer on S_1 the time unit of S_1 appears to be in the ratio $\sqrt{1-\beta^2}$: 1 to that which is described to him as a unit by an observer on S_2 while to an observer on S_2 the time unit of S_2 appears to be in the ratio $\sqrt{1-\beta^2}$: 1 to that which is described to him as a unit by the observer on S_1.

Thus we have the extraordinary conclusion that the time units of the two systems of reference S_1 and S_2, not at rest relatively to each other, are of different lengths in such a way that an observer on either system thinks that the time unit of the other system is greater than his own. It is evident that no simple change of the unit on either system (or both) will bring the units into agreement for observers on both systems. As postulates V and L and T are generally accepted and have not elsewhere led to such strange conclusions it is natural to suppose that the strangeness here is not due to them. In the argument the restricted principle of relativity needs to be used only in so far as it is involved in the conclusion that the units of any two systems of reference S_1 and S_2 are such that the same numerical result is obtained in measuring with the units of S_1 a quantity L_1 and with the units of S_2 a quantity L_2 when the relation of L_1 to S_1 is precisely the same as the relation of L_2 to S_2. But this principle is accepted in the classical mechanics and has not elsewhere led to strange results. The conclusion in postulate M appears to be demanded by the strongest experimental evidence; it is generally accepted; if the strange element in the result concerning units of time is due to this postulate, it appears that we must accept it as being required by such experience as has already been tested with due care. Hence the conclusion seems to be inevitable that the strangeness in our result is due principally to postulate R.

We shall presently see that the same basis of postulates leads to the conclusion that corresponding units of length in the two systems are also different when taken in certain directions. From the transformations of time and space which result from the conclusions thus obtained the whole restricted theory of relativity may be deduced (as is shown in the book mentioned). Therefore this theory depends essentially on the principle of correspondence of units in two systems of reference and on the propositions set forth explicitly in the postulates; and all of these are either generalizations from experiment or statements of laws which have usually been accepted. Hence we conclude: *The restricted theory of relativity may be developed by logical processes from the generalized results of certain experiments together with certain laws which have for a long time been accepted.*

The main result concerning the relation of units of length may be put in the following form:

If two systems of reference S_1 and S_2 move with a relative velocity v and if β is the ratio v/c of v to the velocity c of light as measured on either system, then to an observer on S_1 the unit of length of S_1 along the line of relative motion appears to be in the ratio $\sqrt{1-\beta^2}$: 1 to that which is described to him as a unit by an observer on S_2 while to an observer on S_2 the unit of length of S_2 along the line of relative motion appears to be in the ratio $\sqrt{1-\beta^2}$: 1 to that which is described to him as a unit by the observer on S_1.

These remarkable conclusions concerning units of length in two systems of reference rest on just those postulates which led to the strange results as to the units of time.

What often impresses one as the most remarkable conclusion in the theory of relativity is one which implies that the notion of simultaneity of events happening at different places is indefinite in meaning until some convention is adopted as to how simultaneity is to be determined. In fact, *there is no such thing as absolute simultaneity of events happening at different places.* With respect to the measured time and space of physics we must conclude that time does not run its course independently of space. Measured time and space are indissolubly bound together. The theorem which sets this forth most concretely may be stated in the following way:

Let two systems of reference S_1 and S_2 have an unaccelerated relative velocity v. Let an observer on S_2 place two clocks in the line of relative motion of S_1 and S_2 and adjust them so that they appear to him to mark simultaneously the same time. Then to an observer on S_1 the clock on S_2 which is forward in point of motion appears to be behind in point of time by the amount

$$\frac{v}{c^2} \cdot \frac{d}{\sqrt{1-\beta^2}},$$

where c is the velocity of light, $\beta = v/c$, and d is the distance between the two clocks as measured by the observer on S_1.

By means of the foregoing theorems we may readily obtain the formulae for the celebrated Lorentz transformation of space and time coordinates. (The non-mathematical reader may omit the remainder of this paragraph.) Let two systems of reference S and S' have the relative velocity v in the line l. Let systems of rectangular coordinates be attached to the systems of reference S and S' in such a way that the x-axis of each system is in the line l and that the two x-axes have the same positive direction, and let the y-axis and the z-axis of one system be parallel to the y-axis and z-axis respectively of the other system and have their positive senses in the same directions. Let these two systems of axes coincide at the time zero. Furthermore, for the sake of distinction, denote the space and time coordinates on S by x, y, z, t, and those on S' by x', y', z', t'. Let us suppose that S' moves with respect to S in the direction of increasing values of x. Then it turns out that the foregoing theorems imply the following relations between the two systems of coordinates:

$$t' = \frac{1}{\sqrt{1-\beta^2}}\left(t - \frac{v}{c^2}x\right),$$

$$x' = \frac{1}{\sqrt{1-\beta^2}}(x - vt),$$

$$y' = y,$$

$$z' = z,$$

where $\beta = v/c$ and c is the velocity of light.

The foregoing theorems, or (in more compact language) the foregoing equations of transformation, furnish the effective means for developing the whole of the restricted theory of relativity. Our purpose does not require us to follow that development further in detail. But we may mention a few of the remarkable conclusions which now emerge readily. If two velocities, each of which is less than c, are combined the resultant velocity is also less than c. The mass of a body increases with an increase in its velocity relative to the system on which the mass is measured. The mass of a body at rest appears to be the measure of its internal energy. Mass and energy in general appear to be essentially convertible terms. The velocity of light is a maximum which the velocity of a material body may approach but can never equal or exceed.

The development by Einstein in 1905 of the foregoing restricted theory of relativity led to a fresh analysis of the whole foundations of physics. This was made inevitable by its effective attack upon such fundamental notions as those of length and time and mass and velocity. Einstein himself succeeded in 1915 in greatly extending the range of his theory, developing what has since been called the general theory of relativity. We shall now speak briefly concerning the foundations underlying the latter.

Already in the restricted theory time and space had become essentially blended so that we could no longer speak of a three-dimensional space as separate and apart from the one dimension of time. A sort of combination of the two came into our conception and we began to realize that they can not be disentangled by the measurements of physics. We are forced to consider a four-dimensional continuum of space and time. It is with this space-time extension of four-dimensions that the general theory of relativity has essentially to do; and its problems are intimately connected with the relations of two systems of reference of the generalized sort which this makes necessary. The Lorentz transformation was a great psychological (and even logical) aid in the formation of the new theory.

Let us consider a four-dimensional extension in which space and time are intimately connected and blended so that each point P in these four dimensions represents a definite place A at a definite time t at which A is to be considered. In the course of time a material particle is represented by a succession of these points P. All these points for a given material particle lie on what is called the "world-line" of that particle; and this world-line represents the state of motion (or eventually the state of rest) of the material particle. If two objects come into coincidence at an instant their world-lines have a corresponding intersection. The things which the physicist deals with ultimately are these intersections of world-lines.

In order to deal with them he finds it necessary to introduce certain reference numbers which we may call the coordinates x_1, x_2, x_3, x_4. These numbers change in such a way that their variation along any world-line is continuous and that no two points ever have the same ordered set of four numbers assigned to them. This gives us a very general set of coordinates. It is clear that coordinates can be set up in an immense variety of ways so as to have these few very general properties. One of the first problems in the general theory of relativity is that of the character of the transformation by means of which we can pass from a given choice x_1, x_2, x_3, x_4 of coordinates to a second one $\xi_1, \xi_2, \xi_3, \xi_4$. It is clear that we must have relations of the form

$$\xi_i = f_i(x1, x2, x3, x4)\ ,\ i=1,\ 2,\ 3,\ 4,$$

where the functions f_i of the variables x_1, x_2, x_3, x_4 are rather general functions of these four arguments and are indeed to a large extent arbitrary. Now suppose that the laws of nature are expressed in terms of the coordinates x and also in terms of the coordinates ξ; the question arises as to what relation one ought to expect between these two forms of the law. Now there are no coordinates in nature. These have been inserted by us for our convenience. What is more natural, then, than the demand that we shall formulate our statements of these laws so that they shall have the same form in these two systems of reference, and indeed in all possible systems of reference ? This is precisely one of the fundamental basic requirements upon which Einstein insists. The corresponding principle he has called the principle of covariance. In detailed and precise form, it may be stated somewhat as follows:

PRINCIPLE OF COVARIANCE. *The laws of nature can be (and are to be) expressed in such mathematical form in terms of the space-and-time coordinates* x_1, x_2, x_3, x_4 *that they shall remain invariant under every transformation of the form*

$$\xi_i = f_i(x1, x2, x3, x4)\ ,\ i = 1, 2, 3, 4,$$

where the functions f are subject to the following conditions:

1) *They are (apart from exceptional points or regions of fewer than four dimensions) finite and continuous and indefinitely differentiable;*

2) *They are such that the transformation is uniquely reversible, the inverse transformation having the properties demanded for the direct transformation;*

3) *They are such that in both the transformation and its inverse the fourth variable has the character of a time variable while the other three have the character of space variables.*

This principle demands the attainment of an ideal which is admittedly mathematical in its character. By means of it alone one could not come to grips with phenomena. One needs some additional hypotheses. One of these is to the effect that the restricted theory of relativity is valid in free space, that is, in space free of a gravitational field. The other is the celebrated law of the equivalence of gravitational forces and the apparent forces due to acceleration. This may be set forth as follows:

PRINCIPLE OF EQUIVALENCE. *For an indefinitely small region of the world (that is, a region so small that the variation of gravitation in it in both time and space is negligible) there exists a coordinate system* S_0 *(*X_1, X_2, X_3, X_4*) with respect to which gravitation has no influence either upon the motions of mass particles or upon any other physical phenomena whatsoever.*

Such is the logical basis from which the general theory of relativity proceeds. We can not here follow it in its high enterprise of conquest over the laws of nature. The road (at present and perhaps for a long time to come) can be followed only by one who is willing to give serious and long-continued attention to the study of certain branches of mathematics. In the earlier parts of the argument the reasoning is rather technical and abstruse in character and the general steps are intelligible only to those who have a considerable acquaintance with a certain range of mathematical ideas. After a time the exposition comes down, if not to earth, at least to the solar system and cases begin to appear in which it is possible to find means for choosing between the theory of Newton and that of Einstein.

Three crucial phenomena have been brought to light by means of which to test between the two theories. We shall now speak briefly of each of these.

For a long time astronomers have known that there is a certain forward advance in the perihelion position of the planet Mercury which can not be accounted for on Newton's theory. It amounts to about *42* seconds of angular measure per century. This is well accounted for by Einstein's theory.

Einstein predicted, on the basis of his theory, that a ray of light from a star which is seen apparently close to the edge of the sun would be found to be bent out of a straight path and that the deflection thus caused would turn out to be 1.74 seconds of angular measure, the bending being in such a direction that the star could actually be seen when just behind the edge of the sun. The prediction has been verified with a good degree of precision, observations having been taken at two eclipses of the sun.

A third crucial phenomenon is associated with the vibrations of an atom in a gravitational field. Since the periods of an atom furnish a sort of natural clock, it should give an invariant measure of an interval of time. Proceeding from this hypothesis one concludes that an atom vibrates more slowly on the sun than on the earth, due to the influence of the larger gravitational field of the sun. Hence the lines of the spectrum should be displaced towards the red. For the part of the spectrum usually observed this amounts to about .008 tenth-meters (a tenth-meter = 10^{-10} meters). For a long time there was grave doubt whether this phenomenon is actually existent; but the evidence for its existence now (1933) seems to be conclusive.

Moreover in recent years it has come to be recognized that the stars known as white dwarfs have masses which are comparable with that of the sun, while their radii are much smaller. The companion of Sirius is a star whose radius is about 1/35 of that of our sun. Computation shows that the shift in the lines of the spectrum produced by light passing near this star should be about .30 tenth-meters. This matter was put to the test at Mount Wilson Observatory and an actual shift of .32 tenth-meters was found. One would conclude then that it is now hardly possible to doubt the actual existence of the spectral shift predicted by the Einstein theory.

Whatever may be the final verdict concerning the validity of the theory of relativity as a whole, it has certainly made a fundamental and permanent contribution to astronomy in developing a modification of Newton’s law of gravitation. It has been checked experimentally in three very different ways and is thus established on a rather secure basis. Three such conquests as those just recorded have probably never before been made so nearly simultaneously by a single theory developed from one point of view consistently maintained throughout