CHAPTER XXXIV

ON GEOMETRY

At the same time it will not be forgotten that the physical reality of geometry can not be put in evidence with full clarity unless there is an abstract theory also. . . . Thus, for example, while the term electron may have more than one physical meaning, it is by no means such a protean object as a point or a triangle.
(259) OSWALD VEBLEN

Euclidean space is simply a group. (417) HENRI POINCARÉ

It is only in Euclidean "gravitationless" geometry that integrability obtains. (551)
HERMANN WEYL

The fundamental fact of Euclidean geometry is that the square of the distance between two points is a quadratic form of the relative co-ordinates of the two points (*Pythagoras' Theorem*). *But if we look upon this law as being strictly valid only for the case when these two points are infinitely near, we enter the domain of Riemann's geometry.* (547) HERMANN WEYL

. . . *parallel displacement of a vector must leave unchanged the distance which it determines. Thus, the principle of transference of distances or lengths* which is the basis of metrical geometry, carries with it *a principle of transference of direction;* in other words, **an affine relationship is inherent in metrical space.** (547)HERMANN WEYL

But before dealing with the brain, it is well to distinguish a second characteristic of nervous organization which renders it an organization in levels. (411)HENRI PIÉRON

Section A. Introductory.

The main metrical rule in geometry is the familiar pythagorean theorem. In 1933 this rule is no longer considered as generally valid outside of the euclidean system, as its proof depends on the doubtful postulate of parallels. It is considered as an empirical generalization in which the relative error decreases when the distances become smaller. Indeed the small element of length, *ds*, given by the pythagorean rule is considered convenient and reliable in our exploration of the world.

The pythagorean rule states that in any right triangle, *ABC*, the square of the side opposite the right angle (the hypotenuse) is equal to the sum of the squares of the two other sides (the legs). In symbols, $AB^2 = AC^2 + BC^2$. If we build squares on all three sides of the triangle *ABC* and denote the areas of the squares by C', A', and B' then we have $C' = A' + B'$.

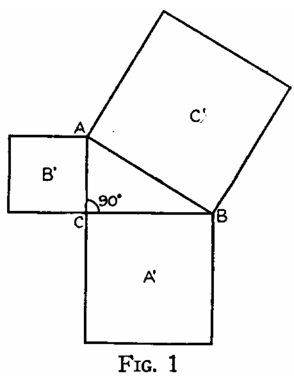

FIG. 1

The above rule is also the main metrical rule for co-ordinate geometry, which gives us the length of the line segment joining any two points. Consider, for example, two points in two dimensions, P_1 and P_2, whose co-ordinates referred to a pair of axes in the plane are (x_1, y_1) and (x_2, y_2). By drawing the lines P_1Q and P_2Q parallel to the X and Y axes respectively, a right triangle P_1QP_2 is formed whose legs P_1Q and P_2Q are equal to x_2-x_1 and y_2-y_1 respectively, whence P_1P_2, the hypotenuse of the right triangle, or the distance s between the points, is equal to $\sqrt{P_1Q^2 + P_2Q^2}$, or $s=\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. If we pass to indefinitely small quantities and choose to deal with differentials we have $ds^2=dx^2+dy^2$ where $dx=x_2$-x_1 and $dy=y_2$-y_1. Usually the physicists treat their differentials as very small quantities and we may do likewise, although this is not precisely what a differential represents.

In three dimensions similar formulae appear; namely, $s^2=x^2+y^2+z^2$ for the distance of a point from the origin and $s=\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ for the distance between two points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2), and also $ds^2=dx^2+dy^2+dz^2$, for the indefinitesimally small distance between two points.

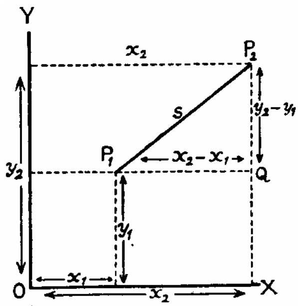

FIG. 2

In referring our geometrical entities to co-ordinate axes, or frames of reference, as they are called, we are interested in the properties of our geometrical entities and not in the accidental characteristics of our frames of reference, or the accidental characteristics of the form of representation we are using. Mathematicians discovered long ago that the form of representation is not of indifference to the results they obtain. Speaking roughly they have discovered that in one form of representation, they obtained characteristics *a, b, c, d, . . . m, n;* in another form, characteristics *a, b, c, d, . . , p, q;* and in still another form, characteristics *a, b, c, d, . . . s, t,* . In cases where direct inspection was possible they find by checking up predicted characteristics, that such characteristics as *a, b, c, d* in our example actually belong to the subject of our analysis, whereas the characteristics *m, n, . . . p, q, . . . s, t, . . . ,* do *not* belong to our subject at all, but *vary* from one form to another depending on the form of representation. Such facts make mathematicians distinguish between characteristics which are *intrinsic,* which actually belong to the subject independently of the form of representation; and those which are *extrinsic,* which do not belong to the subject, but are accidental and vary with the form of representation we happen to use.

If we mix intrinsic and extrinsic characteristics we have a structurally distorted knowledge of our subject. Obviously we are interested in methods by which these two types of characteristics can be separated and distinguished.

Such methods are found in what we call the transformation of co-ordinates, which means the passing from one form of representation to another, from one system of co-ordinates to another which corresponds to translation from one language to another. Obviously those characteristics which are intrinsic to our subject are and must be *independent* of the accidental selection of our form of representation, and therefore should remain unchanged when we pass from one frame of reference to another. Any characteristic which is changed by such a transformation of our systems of reference is clearly an extrinsic characteristic injected by the form of representation and not belonging to our subject; and so the transformation of co-ordinates is precisely the test we need and use.

Let us take for instance the line segment P_1P_2, as in Fig. 3. We may refer P_1P_2 to a system O, or to a system O'. Obviously the length of the line P_1P_2 is independent of the axes of reference used, and the formula for the length of a line is not altered, although the values for the x's and y's are different in the two systems. In other words, the sum of the squares of the differences of the co-ordinates remains invariant. In symbols,

$$s=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(x'_2-x'_1)^2+(y'_2-y'_1)^2}\ .$$

Such expressions, however, as x_1+x_2 or y_1y_2. , are *not* characteristics of our subject but characteristics of the particular frame of reference used, and so are mostly of no interest to us.

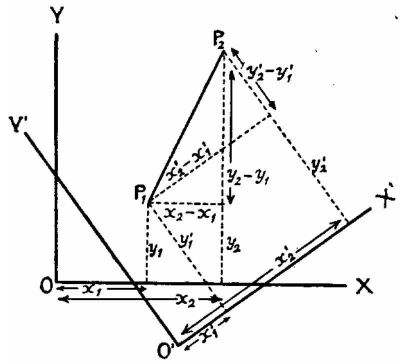

FIG. 3

In such an elementary example as given here we are directly acquainted with our entities, and so we can inspect them directly and check for intrinsic and extrinsic characteristics. But when we deal with geometries of more than three dimensions, such checking becomes difficult, if at all possible, and so new methods have to be invented.

If we wish to eliminate the unit by which we measure our lines, this can be done by using a relation called a *ratio.* Let us, for instance, select 3 points A, B, C, and write the invariant formulae for the distance AB and AC in the form $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and $\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}$

then the *ratio* $$R=\frac{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}{\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}}$$

is independent of our unit of measurement. If, for instance, this ratio $R=1$, we conclude that $AB=AC$, a characteristic which belongs to our lines and which is *independent not only of our system of reference but also of the unit which we have used.*

A great step forward in the formulation of methods which lead to invariant and intrinsic formulations was made in the invention of what is called the vector calculus and its extension in the modern tensor calculus. A few explanations of this principle will be of interest.

A vector is roughly a directed segment of a straight line on which we distinguish the initial and the terminal points. A vector has thus magnitude and direction. In practice we deal with two kinds of entities; some are purely numerical, establishing a specific, mostly asymmetrical, relation and have no direction, as for instance, mass, density, temperature, energy, electrical charge, population, mortality, . These quantities which do not involve direction are called scalar quantities.

Such quantities as velocity, acceleration, electric current, stresses, flow of heat or fluids. , which involve not only magnitude but also a definite direction, are called vector quantities, and have given rise to a special calculus called the vector calculus.

The invention of the vector calculus was a most revolutionary and beneficial structural and methodological step. It was originated independently by Hamilton and Grassmann. The benefits of this method are manifold, but we are interested mainly in but two of them. The first is that vector equations are simpler and fewer in number than co-ordinate equations. The second, and most important, is that the language of vectors is independent of choice of axes, and of frames of reference. It is naturally invariant for any transformations of axes. If axes are needed we can easily and simply introduce them, but we always have means to discriminate between intrinsic and extrinsic characteristics. The modern tensor calculus which made the general theory of Einstein possible is simply an extension of the vector calculus.

The above methodological and structural remarks are of fundamental semantic importance to us in all our affairs. Human life and affairs are never free from linguistic issues. Their role is similar to that of *mathematics,* that is to say, a form of *representation* gives us not only the characteristics which are intrinsic in our subject, but also introduces extrinsic characteristics which do not belong to the subject of our analysis but are due to the particular language we use and its *structure.* The analysis of these linguistic issues is much belated and extremely difficult because of the structural complexity of our language. These issues were discovered first in mathematics because of its clear-cut structural *simplicity;* and it is important that we should be aware of such new and unexpected fundamental semantic problems. We will not enlarge upon this phase of the problem here, except to mention that the whole of the present work, which uses a different language, of a different *structure,* already shows the usefulness of the new method. Sometimes we discover new characteristics, and sometimes we are led to emphasize characteristics which are known but have not yet been sufficiently analysed.

To carry our linguistic analogy further, we may take, for instance, the statement, ‘knowledge is useful’. We could translate this statement into any other language and it would preserve its meaning. But if we make the statement, ‘knowledge is a word which has six consonants and three vowels’, such a statement may be false when translated into another language. Mathematics, being a language, has difficulties similar to ordinary language, but in mathematics it is often much more difficult to separate from other statements those which are purely about the language used. The so-called tensor calculus attempts to perform this last task.

The tensor calculus is an extension of the vector calculus, which has become famous since Einstein. It gives us formulations independent of any special frame of reference. In using it we are automatically prevented from ascribing to the events around us characteristics which do not belong to them. The tensor equations give us absolute formulations, absolute being understood as relative, no matter to what. Obviously the only language fit to express the 'laws of nature' should be independent of the particular point of view or language of some observer. It should give us formulations invariant for any and all systems of reference, although we might use preferred systems of reference, as, for instance, the principal axes of an ellipse, without any danger. The reader should not miss the point that such an ideal should be considered as the highest ideal in science. It is the mathematical species of a theory of 'universal agreement'. The above *sounds* simple and innocent; but, when actually applied, plays havoc with most of our old 'universal laws'. These laws do not survive this important and uniquely valid test, and so become mere local gossip instead of being the 'universal laws' that they claim to be. We will return to the structural problem of invariant formulations later. At present we must explain some other simple considerations.

On any surface we need two numbers or 'co-ordinates' to specify the position of a point, and so a surface is called a two-dimensional manifold. Points in three-dimensional manifolds require three numbers; points in four-dimensional manifolds four numbers; and similarly for any number of dimensions.

For our purpose, it is enough to speak in two dimensions, as our statements can easily be generalized to any number of dimensions. If we want to localize a point on a surface it is enough to divide the surface into *meshes* by any two line-systems which cross each other. By labeling the lines of each system with consecutive numbers, two numbers, one from each system, will specify a particular mesh. If the meshes are small enough we will be able to locate any point accurately.

These specifying labels or numbers require that we know what kind of mesh we are using. *Distances* between points are *independent* of mesh systems.

For the above reasons it is important to have more data about the mesh system we are using, which means that we have formulae which express the *distance* between two points, which is independent of the mesh systems, in terms of the mesh system.

We have already seen, in our study of the differential calculus, that, as a rule, it is simpler to deal with very short distances, and that it is easy to pass to larger distances by the process of integration. As yet we have used only plane rectangular systems of meshes in our illustrations, but this restriction is not necessary. If we use oblique co-ordinates (Fig. 4), the formula for the elemental distance is $ds^2=dx_1^2-2kdx_1dx_2+dx_2^2$, where k=cosine of the angle between the lines of partition.

The polar co-ordinates (Fig. 5) of the point P are the distance $r=OP$, of the point from the origin O, and the angle, $\theta=\angle XOP$, between the line OP

and the axis OX. The formula for the elemental distance in polar co-ordinates is $ds^2=dr^2+r^2d\theta^2$

Fig. 6 shows the co-ordinates frequently used by geographers; namely, geographic longitude and latitude, where the distance $ds^2=d\beta^2+\cos^2\beta d\lambda^2$.

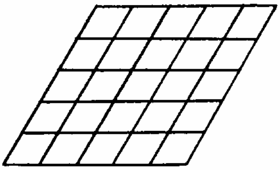

Fig. 4

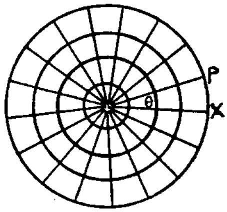

Fig. 5

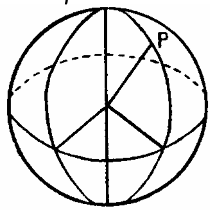

Fig. 6

It should be noticed that these formulae for different systems of co-ordinates are *different.* To make it still more obvious to ocular inspection, we will tabulate them in one lettering, thus:

$ds^2 = dx_1^2+dx_2^2$	for rectangular systems	
$ds^2 =dx_1^2+x_1^2dx_2^2$	for polar systems	(1)
$ds^2 =dx_1^2-2kdx_1dx_2+dx_2^2$	for oblique systems	
$ds^2 =dx_1^2+\cos^2x_1dx_2^2$	for latitude and longitude systems.[1]	

It must be noticed that the values for the variables are not equal in these different equations. It is not necessary for the reader to know in detail how these formulae are obtained, but it is necessary to see that they are *different,* that they have different structure. The numbers of different co-ordinate systems we can use are infinite, but in practice we use only a few well-known types. There are also definite and simple formulae for passing from one system of co-ordinates to another.

We should not assume that in practice we always know what system of co-ordinates we are employing. For instance, before we learned that our earth is ‘round’, we did not know whether in our measurements we were employing the flat co-ordinates of a plane or spherical co-ordinates. We made some measurements and then we had to discover what kind of formulae would fit these measurements.

To find out what kind of co-ordinate system we are using, we select two points, let us say (x_1, x_2) and (x_1+dx,x_2+dx) very close together, make our measurements of ds, and then test our ds to find which formula it fits. If we find for instance that our ds^2 is always equal to $dx_1^2+dx_2^2$ we may assume for simplicity and our purpose that our co-ordinate system is plane and rectangular.

If our measurements fit any of the first three formulae (1), we may assume for simplicity and our purpose, that we are dealing with a plane surface, as each of these systems belongs to the plane. But if we find that the actual measurements of ds^2 are such that they never fit these first three formulae, but only the fourth one, we know, that our surface is not plane but curved like a sphere. Try as we may, we shall be unable to build on a plane any co-ordinate system which will fit the last formula. Thus we arrive at an important conclusion; namely, that from *measurements* we have a *structural* hint as to the *kind of world* we are in.

Section B. On the notion of the 'Internal Theory of Surfaces'.

Let us imagine some two-dimensional beings confined to their surface and unable to have a look at that surface from our third dimension. For them our third dimension would be 'unthinkable' and therefore the surface of a sphere like our earth which is curved in the third dimension would also be 'unthinkable' or 'beyond them'. Should they conduct some measurements in their 'world' and find that these measurements did not fit any of the first three formulae but only the fourth, they would have to reconstruct radically their 'world conception' and conclude that their world was a spherical surface. Our own situation does not differ radically from the situation of the inhabitants of this hypothetical two-dimensional world.

If we find ourselves so restricted as not to know whether we are finally dealing with a flat or spherical surface, we can select a point *O* and with a definite radius *R* describe from this point a circle *ABC*. Then we can measure the circumference of this circle. Now we know from geometry that in *the plane* the circumference of the circle $L=2\pi R$ where *R is* the radius of the circle and $\pi=3.1415\ldots$ If our surface is flat (*ABCD*), our measurement of *L* and *R* will satisfy the relation expressed in the formula. But if the surface is curved, our $R=OA$ will be larger than $R'=AO'$, and we shall find that our π is not 3.1415 . . . , but smaller. We see once more that the *metrical properties of our world throw some light on its structural character.*

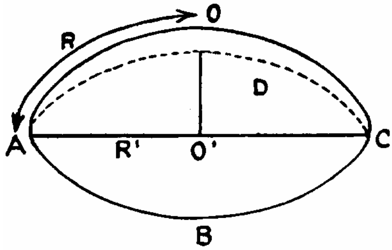

FIG. 7

We should notice also that the curvature of a two-dimensional surface is in the third dimension and that it is the means of giving us data about the surface without our leaving the surface and going into a third dimension. It is easy to convince oneself about these facts by taking 12 wires or strings of equal length and constructing the figure shown in Fig. 8. If we build it on a flat surface the 12 equal wires will fit exactly. But if we try this experiment on a curved surface, for instance on a pillow-, or saddle-shaped surface, the last closing wire will not fit, and will be too short or too long depending on the kind of surfaces we have.

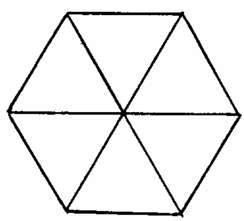

FIG. 8

The formulae (1) have been generalized to

$$ds^2=g_{11}dx_1^2+2g_{12}dx_1dx_2+g_{22}dx_2^2 \qquad (2)$$

for two dimensions and to

$$ds^2 =g_{11}dx_1^2+g_{22}dx_2^2+g_{33}dx_3^2 +g_{44}dx_4^2+2g_{12}dx_1dx_2$$
$$+2g_{13}dx_1dx_3+2g_{14}dx_1dx_4+2g_{23}dx_2dx_3+2g_{24}dx_2dx_4 +2g_{34}dx_3dx_4 \qquad (3)$$

for four dimensions. It is easy to see that $ds^2=dx_1^2+dx_2^2$ is obtained from (2) by taking $g_{11}=1$; $g_{12}=0$ and $g_{22}=1$. This applies to formula (3) out of which we can have any of the other formulae by equating some of the *g's* to zeros, or to one or to other values. Formula (3) is called ‘the generalized pythagorean rule’, of which the ordinary form as given previously is only a particular case. We see, by comparing the formulae (1) with (2) and (3), that these *g's* are not equal for different systems of co-ordinates, and that they are factors in measure-determination which represent the geometry of the surface considered. It is customary to write the above formulae in an abbreviated form: thus $ds^2=\Sigma\Sigma g_{mn}dx_mdx_n$, where we give to m and n the values 1, 2, 3, 4, or (m, n = 1, 2, 3, 4) and where the symbol Σ means summation.

We will now explain briefly the above generalizations and the meaning of the *g's* given in the expressions.

In the beginning of the nineteenth century the mathematician Gauss formulated the *internal* theory of surfaces without reference to the plenum in which they are embedded. This theory perhaps is and will remain a model on which all theories should be built. He introduced also a new kind of co-ordinates which have become of paramount importance, and which since Einstein are called gaussian co-ordinates. Gauss investigated the theory of surfaces, which are in general curved, embedded in three-dimensional ‘space’. In 1854 the great mathematician Riemann generalized the two-dimensional gaussian theory to a continuous manifold of any number of dimensions. Historically, both Gauss and Riemann can be considered as the precursors of Einstein.

Let us imagine a surveyor to have the task of mapping a thickly wooded hilly region. Because of the conditions of his work, he can not use optical instruments, and he has no ‘straight lines’ to deal with. So euclidean geometry will, in general, not be applicable to the region as a whole. It can be assumed, however, that euclidean geometry may be applied to *very small* regions which can be considered flat. What we know already about the differential and integral calculus shows us that such approximations, when taken on a very small scale, are perfectly reliable and justifiable.

The surveyor would lay out on his ground a network of smoothly curving lines, in two families, an *X* family and a *Y* family. (Fig. 9.) All the curves of the *X* family would intersect all the curves of the *Y* family but no *X* curve would intersect another *X* curve, nor a *Y* curve another *Y* curve.

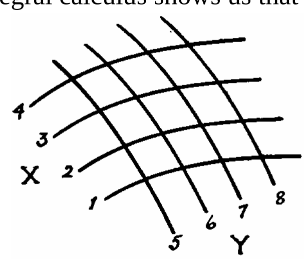

Fig. 9

Let us take the surveyor's network and label the curves by consecutive numbers in each family. The essential point is that these numbers, (let us call them the *X* and *Y* numbers) do *not* represent either lengths, or angles or other measurable quantities, but are simply labels for the curves, much as when we label streets by numbers.

But such numbering does not lead us far. We must introduce some *measure relations.* We have at our disposal a measuring chain and the *arbitrary* meshes of the network which we have introduced. The next step is to measure the small meshes one after another and plot them on our map. When this is done we have a complete map similar in structure to our region. Because of the smallness of the meshes we can consider them as small parallelograms, and such parallelograms can be defined by the lengths of two adjacent sides and one angle.

We may, however, proceed differently, as shown on Fig. 10.

Let us select one mesh, for instance the one bounded by the curves, 3 and 4 and by the curves 7 and 8. Let us consider a point *P* within the mesh, and let us denote its distance from the point *O* (x=3,y=7) by *s*. This distance could be directly measured. Let us draw from the point *P* parallels to our mesh lines and label the intersections with mesh lines by *A* and *B,* respectively. Let us also draw *PC* perpendicular to the *x* axis.

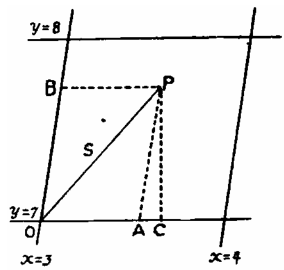

FIG. 10

The points *A* and *B* then also have numbers or labels or gaussian co-ordinates in our network. The co-ordinate of *A* may be determined by measuring the side of the parallelogram on which *A* lies and the distance of *A* from *O*. We can regard the relation called the *ratio* of these two lengths as the *increase* of the *x* co-ordinate of *A* towards *O*. We shall denote this increase itself by *x*, choosing *O* as the origin of the gaussian co-ordinates. Similarly, we determine the gaussian co-ordinate of *y* of *B* as the ratio in which *B* cuts the corresponding side. We see that these two *ratios,* which for brevity we call *x* and *y*, are the co-ordinates of our point *P*.

As *x* and *y* are *ratios* they of course do not give us the lengths of *OA* and *OB* but the lengths are given, for example, by *ax,* and *by,* where *a* and *b* are definite numbers, to be found by further measurements. If we move the point *P* about, its gaussian co-ordinates change but the *numbers a* and *b* which give the ratio of the gaussian co-ordinates to the true lengths *remain unchanged.*

We find the length, *s*, which is the distance of the point *P* from *O*. from the right triangle *OPC* by the pythagorean rule: $s^2=OP^2=0C^2+CP^2$. But $OC=OA+AC$ and therefore by substituting and squaring we have $s^2=AO^2+2OA{\bullet}AC+AC^2+CP^2$. From the right triangle *APC* we have $AC^2+CP^2=AP^2$, whence substituting again we have $s^2=OA^2+2OA{\bullet}AC +AP^2$. But $0A=ax$, $AP=OB=by$, and as *AC* is the projection of $AP=by$, it also has *a fixed ratio* to it, whence we may put $AC=cy$, and so we obtain the important formula $s^2=a^2x^2+2acxy+b^2y^2$, in which *a, b, c* are *ratios* given *by fixed numbers.* Usually this formula is represented differently, a^2 is designated *by* g_{11}, *ac by* g_{12}, and b^2 *by* g_{22}; whence our formula becomes $s^2=g_{11}x^2+2g_{12}xy+g_{22}y^2$ in which the numbers 11, 12 and 22 are simply ordering labels without quantitative values, mere subscripts, labels, indices. , which indicate that the different *g*'s have different values. We see that the above formula is the one which was given previously by (2).

The *g*'s with different labels serve just as sides or angles for the determination of the actual sizes of the parallelograms and we call them the *factors* of the measure *determination.* They may have different values from mesh to mesh, but if they are known for every mesh, then, by the last formula, the true distance of an arbitrary point *P*, within an arbitrary mesh from the origin can be calculated.[2]

The procedure by which we can locate any point on the surface is simple. If our point *P* is between the two curves $x=3$ and $x=4$ we can draw nine curves between these two curves and label them 3,1; 3,2; . . .; 3,9. If *P* now lies between curves 3,1 and 3,2 we can draw nine curves between these two curves and label them 3,11; 3,12; . . .; 3,19, . We could do similarly with the *y* curves and in this way we would succeed in assigning to any point as accurate a pair of numerical labels as we pleased, and so finally have the gaussian co-ordinates of any point. We used nine curves simply to get the very convenient decimal method of labeling. The Cartesian co-ordinate systems which we use in plane geometry obviously represent only *special cases* of gaussian systems.

As we have already seen, our *g*'s are *ratios,* and so represent numbers. Such numbers may be regarded as tensors of zero rank for convenience of the mathematical treatment; and the quantities g_{xx}, g_{xy}, g_{yy}, may be treated as components of a tensor. Since this tensor determines the measure relations in any particular region, it is called the *metric fundamental tensor.* Its value must be given for the region in which we want to make our calculations. It determines the full geometry of the surface in a given region; and, conversely, we can also determine the fundamental tensor in a given region from measurements made in that region, without any previous knowledge of how our curved surface is embedded in ‘space’ at the place in question. The fundamental tensor in general varies continuously from place to place, and so every geometric manifold may be regarded as the field of its metric fundamental tensor.

Purely mathematical investigations show that the fundamental tensor defines a number called the ‘Riemann scalar’, which is completely independent of the co-ordinate system and leads to the definition of the *curvature tensor,* which can be connected with the ‘matter tensor’.[3]

The main importance of the introduction of such arbitrary curves is to produce formulae for the surfaces which remain unaltered for a change of the gaussian co-ordinates—in other words, which remain invariant. This was achieved by the introduction of the relations called *ratios* which are pure numbers, and so the geometry of surfaces becomes a theory of invariants of a very general type.

On curved surfaces there are in general no straight lines—there are *shortest lines*, which are called 'geodetic lines'. To find them, we divide any arbitrary lines joining two points into small elements, which we measure, and select the line for which the sum of these elements is less than for any other line between the two points.[*] Analytically we can calculate them, when the *g*'s are given, by the aid of the generalized pythagorean theorem. The geodetic lines, and also the curvature, are given by invariant formulae, which represent intrinsic characteristics of the surface, independent of any co-ordinates. All higher invariants are obtained from these invariants.[4]

We shall not attempt to give an explanation of the tensor calculus, as at present there is no elementary means of presenting a brief explanation; short of a small volume—at least the writer does not know of any.[5]

The name 'tensor' originally came from the Latin word *tendere=to* stretch, whence tensio=tension. Nowadays, however, it is used in a more general way; namely, to express the *relation* of one vector to another, and not necessarily to imply stress or tension. As an example, we can give the representation for stresses occurring in elastic bodies, which originally led to the name.[6]

As we have already seen, when we deal with *relations* of vectors our expressions become additionally independent of units. Such equations, independent of the measure code, are called tensor equations.[7]

As we are interested in equations which are invariant under arbitrary transformations, certain functions, called tensors, are defined, with respect to any system of co-ordinates by a number of functions of these co-ordinates, called the components of the tensor, from which we can calculate them for any new-system of co-ordinates. If two tensors of one kind are equal in one system, they will be equal in any other system. If the components vanish in one system, they vanish in all systems. Such equations express conditions which are independent of the choice of co-ordinates. By the study of structural laws of the formation of tensors we acquire means of formulating structural laws of nature in generally invariant forms. Obviously, such methods and language are uniquely appropriate for physics and the formulations of the laws of nature. If a law cannot be formulated in some such form, there must be something wrong with the formulation and it needs revision.

The tensor calculus is also peculiarly fitted to describe processes in a *plenum*. We do not use it to describe the metrical conditions but to describe the *field* which expresses the physical states in a metrical plenum.

Eddington gives an excellent example of the fact that it is definitely necessary to look into the way we build up our formulae (structure) and the method of handling them.

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[*] More generally, the geodetic represents a track of minimum or maximum interval-length between two distant events, either of them being unique (one-valued) in a given case.

The problem is to determine whether a particular kind of space-time is possible. We must investigate the different g's which give us different kinds of space-time, and not those which distinguish different kinds of mesh systems in one space-time. This means that our formulae must not be altered in any way if we change the mesh system.

The above condition makes an extraordinarily simple test of laws that have been or may be suggested. Among others, Newton's law is swept away. How this happens can be shown in two dimensions.

If in one mesh system (x,y) we have $ds^2=g_{11}dx^2+2g_{12}dxdy+g_{22}dy^2$, and in another system (x',y') $ds^2=g'_{11}dx'^2+2g'_{12}dx'dy'+g'_{22}dy'^2$, one law must be satisfied if the unaccented letters are replaced by accented letters. Let us suppose that the law $g_{11}=g_{22}$ is assumed. We change the mesh system, for instance, by spacing the y lines twice as far apart, that is, we take $y'=y/2$ and keep $x'=x$. Then $ds^2=g_{11}dx^2+2g_{12}dxdy+g_{22}dy^2=g_{11}dx'^2+4g_{12}dx'dy'+ 4g_{22}dy'^2$. We see that $g'_{11}=g_{11}$ and $g'_{22}=4g_{22}$. Whence if g_{11} is taken equal to g_{22}, g'_{11} *cannot* be equal to g'_{22}.

A few examples would convince us that it is extremely easy to change a formula entirely by the mere change of mesh systems. It seems unnecessary to emphasize the fact that 'universal laws', to be 'universal', should not depend structurally to such an extent on the accidental and, after all, unimportant, choice of reference systems.[8]

To remedy such a state of affairs, impossible in mature science, the tensor calculus was invented. The whole general theory of Einstein seems to demand that the equations of physics should ultimately be expressed in tensor forms; in other words, that 'universal laws' should cease to be 'local gossip', a demand which must be granted, and *on this point* the Einstein theory is beyond criticism and is an epochal methodological advance of an irreversible structural linguistic character.

Section C. Space-time.

In dealing with co-ordinate systems we have heretofore used them to represent only 'spatial' entities, spreads of different dimensions. It is desirable to become acquainted with a different use of co-ordinates, in which one of them will represent 'time'. The last use is just as simple as the former, but the graphs which we obtain are different.

FIG.-11.

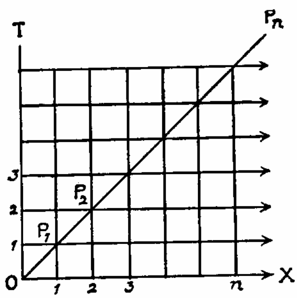

FIG. 12

Let us take the simplest example, of a point P moving uniformly along a straight line OX with the velocity of one inch per second. We could represent its movement in one dimension, as in Fig. 11, and *say* that our point P is at P_1 after one second (t=1), at P_2 after two seconds (t=2) . ; at the point P_n, after n seconds ($t=n$).

But we could also represent this movement in a different way. We could choose two mutually perpendicular axes *OX* and *OT* as in Fig. 12, *OX* representing the 'spatial' actual direction of the movement and *OT,* which we have heretofore used to represent a second 'spatial' co-ordinate, now representing the 'time' co-ordinate.

We would lay off on the *X* axis our inches, 1, 2, 3, . . . *n,* and on the *T* axis our seconds 1, 2, 3, . . . *n.* In our two-dimensional *space-time* our point *P* would be at the point *O* (x=0, t=0). After one second it would be at the point P_1 (x=1,t=1), after two seconds at the point P_2 (x=2,t=2) . ; after *n* seconds at the point P_n (x=n,t=n). We see that the position of our *point P* in two-dimensional *space-time* would be represented by a series of points each given by two data: one 'spatial', the other a corresponding 'time'. If the intervals are taken indefinitely small, in the limit our *'moving' point would be represented by a static line inclined to the X axis.* We could then speak either of our 'moving' point, or else *not use* the term 'moving' but speak of infinitely many *static points,* each given by two numbers, one representing a distance, the other 'time'. Our 'moving' *point* would become a *static world-line.* The reader should notice that in this case we have structurally changed our *language* from dynamic to static, and raised the dimension. Our mathematical 'moving' 'point', which had no dimension in our one-dimensional 'space', is in our two-dimensional *space-time* represented by a *static* one-dimensional *line.*

In this example we had uniform translation. We did not introduce acceleration. The distances were proportional to the 'times', hence our line was 'straight' and inclined to the *X* axis at a constant angle.

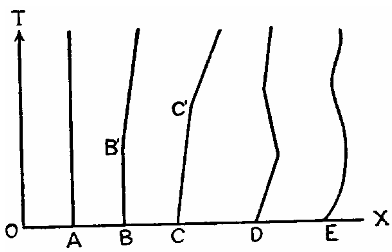

FIG. 13

Using such space-time representation we see that a point when it is not 'moving', but is stationary, is represented by a line parallel to the 'time' axis *T.* as shown at *A* on Fig. 13. Our point *A* is getting older, so to speak, but does not 'move'. In the next case, the point *B* does not 'move' until it is some seconds old, when at *B′* it begins to 'move' with constant velocity. Point *C* 'moves' in the beginning at one constant velocity until *C′* where it acquires a certain different velocity and the direction changes.

In Fig. 13, *D* represents a point experiencing a series of sudden changes of velocities. The graph is a succession of short straight lines forming a broken line or open polygon. As the changes of velocity occur more and more frequently the sides of our polygon become smaller and smaller; and in the limit, as the changes of velocity become continuous, our broken line becomes a smooth curve *E.*

Motion with continuously changing velocity is called accelerated or retarded motion. The rate of change of velocity is called acceleration and is represented by the second derivative of the distance with respect to the 'time'; symbolically, $A=dv/dt=d^2s/dt^2$.

It is important to notice that in space-time an *accelerated* motion is represented by a *curved* line. In uniform (constant velocity) motion the distances are proportional to the 'times', and the line is straight and its equation is of the first degree. In accelerated motion the distances are not proportional to the 'times', the lines are curved and the 'time' element dt enters in the second degree at least; namely, as dt^2.

For example, let us study the graph of the motion represented by the equation $x=At^2/2$ which means that the distance x is proportional to the square of the 'time'.

Let *OX* (Fig. 14) be the 'spatial' axis, and *OT* the 'temporal' axis. We lay off on our *T* axis equally-spaced points, representing the seconds 1, 2, 3, 4. , and calculate the distances x for each of these values from the equation $x=At^2/2$ where A represents a constant acceleration.

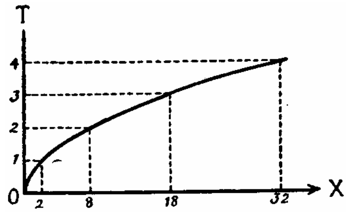

FIG. 14

Let us assume that the constant acceleration A is given as 4 metres per second per second. The equation $x=4t^2/2$ becomes $x=2t^2$. Corresponding to the values t=0, 1, 2, 3, 4. , we have the values x = 0, 2, 8, 18, 32 , . If we plot these points, and assume that the change is continuous, we may join the points by a continuous *curve*, which represents the motion of the point as a *curved* world-line.

Similarly, in three-dimensional space-time, a point moving uniformly in the *plane XY* would be represented in the *plane XY* by the *line AB*, and in three-dimensional space-time by the static line *AB'*, where the 'times' are proportional to the distance.

As we have already seen, non-rectilinear motion may be considered as accelerated motion. We will generalize the above to the case where any *curved* path is traversed with *constant* velocity. In this case the direction of the velocity is changed. If we take the motion of a point which describes a circular orbit with *constant* velocity, it is easy to find its accelerations which is called in this case centripetal.

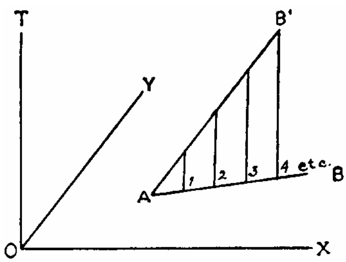

FIG. 15

Let us consider a point *P*, moving in a circular orbit with a *constant* velocity *v*, as given in Fig. 16. If at a certain 'time' it is at *A*, after a short interval *t*, it will be at *B*. The direction of the velocity will be changed from *AA'* to *BB'*.

If we construct the triangle *DCE* by drawing *CD* parallel and equal to *AA'*, and *CE* parallel and equal to *BB'*, we see that the angle $\angle DCE$ is equal to the angle $\angle A'A''B'$ because the sides are parallel, and it is also equal to the angle $\angle AOB$ whose sides are perpendicular to *AA'* and *BB'*. The triangles *ABO* and *CDE* are similar because they are isosceles and the angles between the equal sides are equal. Clearly the side

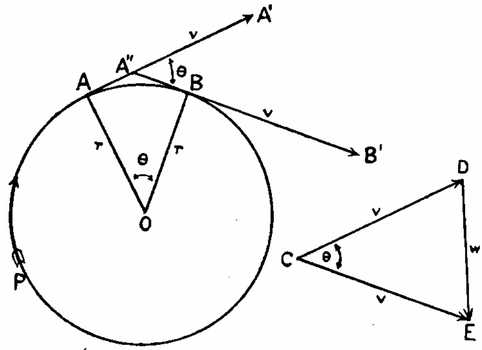

Fig. 16

$DE=w$, represents the supplementary velocity which transforms *AA'* into *BB'*. We know that in similar triangles the sides are proportional so we can write $DE/CD = AB/OA$. By inspection of our figures we see that $DE=w$; $CD=v$; $OA=r$, the radius of the circle. The chord *AB* may be taken as the arc *AB* of the circle, provided the 'time'-interval is taken sufficiently small. Let us write chord $AB=s$. We have $w/v = s/r$ or $w = sv/r$. If we divide both sides of our equation by t we have $w/t=sv/tr$. But $w/t=A$, the acceleration, and $s/t=v$ hence $A=v^2/r$. In words, the centripetal acceleration is equal to the square of the velocity in the circle divided by the radius.

The above formula is of structural importance because it is the foundation for the empirical proof of Newton's law of gravitation. For our purpose it is important for other reasons, to be stated later.

There are two more diagrams which should be considered, in this connection. Fig. 17 represents the plane circular motion of a point *P* whose orbit in the plane *XY* is the circle *PAB*. In three-dimensional space-time the plane circular orbit of motion would be represented by the static cylindrical helix (or screw-line) with axis parallel to the 'time' axis *T*. (Fig. 17.) We should note that the motion is *dynamically circular* in the *XY* plane, yet a three-dimensional space-time represensation gives us a stationary helix.

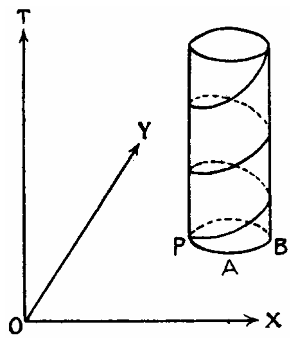

Fig. 17

Similarly for vibrational movements, which could be represented in one dimension by to-and-fro movements on the *X* axis from *A* to *B* and from *B* to *A*. (Fig. 18.) If we introduce our space-time form of representation by introducing the *T* axis, our vibrational world-line would be represented structurally by a wave-line along the *T* axis. In particular, if the vibrational motion is simply harmonic, a proper choice of the 'time' unit makes the wave-line a sine curve.[9]

Becoming thoroughly familiar with these few simple examples takes away a great deal of mystery from the Minkowski-Einstein and the new quantum world. We see that after all there is nothing extraordinary in the fact that in languages of different structures we get different forms of representations and pictures, and that in a world where accelerations abound we may very profitably use the term ‘curved’.

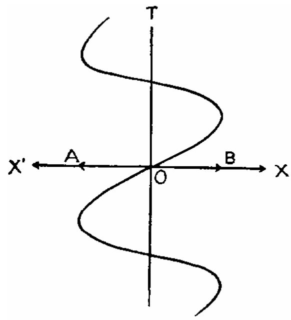

FIG. 18

When we come to speak about the Einstein theory, the four-dimensional space-time world of Minkowski, and the new quantum mechanics we shall have considerable use for these few notions and illustrations.

Section D. The application of geometrical notions to cerebral localization.

In the present work we are dealing in the main with structure and the adjustment of the structure of our languages to empirical structures, and at this point it will be of use to suggest some of the consequences which follow from what has been said.

The question of cerebral localization is a difficult and vital problem. In former days it was supposed that the brain had individualized centres with strictly defined functions. Attempts were made to ascribe to definite cerebral parts definite functions such as ‘memory’, ‘intelligence’, ‘morality’, ‘talents’, . In the meantime, experimental facts disproved such structural views, and as a reaction another tendency appeared; namely, to deny any localization.

Modern researches show unmistakably that both of these extreme tendencies are at variance with experimental structural facts. It appears that the lower centres play a more important role according as the terminal, or higher centres, are less developed and that there is considerable variability, at least in man, not only from the morphological and histological aspects but also from the functional aspect. It was found impossible to generalize from the particular development of centralization and functional distribution in one species to the distribution in another species. Localization may vary even in one individual under different circumstances.[10] Metabolism, and slight disturbances in the functioning of a neuron, were also found to have a most far-reaching influence, shown in its relations to other groups of neurons. The problems of localization are far too complex to attempt even an account of them, the more so since the reader will find excellent accounts of them in the large literature on the subject. The general conclusion reached by practically all investigators is that some localization of nervous function does exist, yet it has a certain variability which depends on an enormous number of factors.

The methods explained in this chapter will enable us to suggest a method by which we can orient ourselves in the bewildering complexity of the functioning of the nervous system.

One of the main difficulties is that the structure of this world is such, that it is made up of absolute individuals, each with unique relationship toward environment (in the broadest sense); and we have to speak about it in terms of generalities. ‘Laws’. , formulated in the old *two-valued ways,* can never account adequately for the facts at hand, being only approximations. The mathematical methods which have already been explained give us at once a great advantage. We have seen that if we have a function, $y=f(x)$, let us say, and take the graph of this function, to every point of the graph there corresponds a pair of values x and y. We have seen also that each of the four quadrants I, II, III, IV has a characteristic pair of signs. In quadrant I, both x and y are positive; in II, x is negative and y positive; in III, both x and y are negative, and finally, in IV, x is positive and y negative. We can easily see that the value of the variables may be thought of as variable conditions different for each individual, and that definite *localizations* correspond to them. In our example we had to do with a function of one independent variable, and we had a one dimensional line, curved in two dimensions. When we had a function of two independent variables we had a surface, which in general was curved in a third dimension. By analogy we may pass to any number of dimensions, where by dimension we do not mean anything mysterious, but roughly the number of variables involved in the problem.

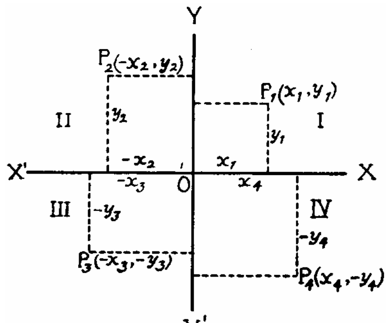

FIG. 19

We see that if we think of the activity of the nervous system in terms of a mathematical function with an enormous number of variables, we shall not only have place for the uniqueness of each individual, determined by the value of the variables and the character of the function, but that this would also imply a *localization,* which is *permanent* in a given individual at a given ‘time’; which again implies the totality of ‘circumstances’, . Our function would be $N=f(x_1, x_2, x_3, \ldots x_n)$.

In fact it is hard to see how it is possible to analyse the activities of the nervous system in any other way. The facts are, that every organism is an individual, distinct and different from others, and so we must have means to take this individuality into account. Different values for different variables take care of this point. Similarities are accounted for by the general structural character of the functions. For instance, any quadratic equation with two unknowns gives us a conic section. An equation of the type $y^2=ax$ represents a parabola, the graph of any equation of the form $xy=a$ represents a hyperbola , . For every definite set of values of our variables the implied localization is also definite, which corresponds to the fact that in a given individual at a given ‘time’. , the localization is definite. One value for the whole function can be reached by giving different sets of values to the different variables. For instance, in the function z=5x-2y-1, if x=1 and y=1, then z=2; but we can have z=2 by taking x=2 and y=3.5. Or if one of the conditions be non-existent, which means that the value of one of our variables is zero, for instance, x = 0, we still could have the value z = 2 by taking y =-1.5. This fact accounts for the *many-to-one* correspondences of causal factors, typified, for instance, by sunshine *or* cod liver oil producing a similar effect.

It should be understood that in what is said here, the numerical values do not matter. In most of the cases we are not advanced enough to be able to deal with such numerical values. What is to be emphasized is the structure of the language we use. The method should enable us, instead of dealing with generalizations in the old language, which somewhere have to be contradicted, to use a language of mathematical structure which shall account for the facts and leave room for the great individual varieties of organisms in structure and function.

After all, we should not be surprised that the theory of functions and language of functions is structurally appropriate in expressing, and so in understanding, the *functioning* of the nervous system, or any other system. Personally I have benefitted greatly through this method; and many baffling structural complexities have been much simplified.

Structurally, when we use the language of functions, variables. , we automatically introduce *extensional* structure, as already explained, and we have at our disposal methods of translation of different orders of abstractions— dynamic into static, and vice versa—which is a neurological structural necessity for being rational and sane. And surely science should try to be rational. It should be stressed again that in our problem numerical values matter very little, but structure and method, for the many reasons already explained, are of paramount importance. Perhaps even the value of numbers is due mainly to the structural fact that it has forced upon us extensional and relational methods. It is the only language which is in accordance with the structure and functioning of the nervous system, and so helps to co-ordinate these activities instead of disorganizing them.

That these simple structural dependences have been discovered so late is really astonishing. The only explanation I can give of this is that we have been so engrossed in generalizing and generalizations that we lost sight of the fact that in life we deal structurally with *absolute individuals*, and that the only *language* which preserves the *extensional structural individuality* for its elements is found in mathematics—specifically, in numbers.

It may be that a study of mathematical structure and the psycho-logics of mathematics will give results of unparalleled human values, particularly for our sanity. The problems of sanity are problems of adjustment, and no means of adjustment should be disregarded. It may also be that the main importance of mathematics will be found some day to be more in the mathematical *methods* and structure which it has originated, methods forced upon the mathematician by the relational character of the entities he has to deal with, than in the possible combinations of these entities themselves.

At any rate, we must sadly admit that the problems of mathematical methods and structure and the psycho-logical values of mathematics have so far received very little attention, since we have failed to realize their human importance. In the future this problem will be further, and thoroughly, investigated.