CHAPTER XXXIII

ON LINEARITY

The conception of linear transformation thus plays the same part in affine geometry as congruence plays in general geometry; hence its fundamental importance. (547) HERMANN WEYL

It is instructive to compare the mathematical apparatus of quantum theory with that of the theory of relativity. In both cases there is an application of the theory of linear algebras. (215) W. HEISENBERG

This "perturbation theory" is the complete counterpart of that of classical mechanics, except that it is simpler because in undulatory mechanics we are always in the domain of *linear* relations. (466) E. SCHRÖDINGER

As a result of experimental research on association, in 1904, I was led to show the complexity of the factors governing evocation.... And I have often insisted since then on this essential idea, in opposition to the simple schema of linear associative connection. (411) HENRI PIÉRON

We have already had several occasions to mention the 'plus' or additive issues as connected with linearity. This problem is of structural and linguistic as well as empirical and psycho-logical semantic importance. It is sufficient for our purpose at present that we should notice two facts; namely, (1) That in one dimension, linearity expresses the *relation* of proportionality; (2) That the problems of linearity are dependent on the *relation* of additivity.

The structural notion of additivity is of great antiquity. Being the simplest of such notions, it naturally originated very early in our history. The earliest records show that the Babylonians and the Egyptians used the additive principle in their notation. Our primitive ancestors, long before any records were written, had similar structural conditions present, open for investigation and reflection, that we have today. That this was the case is not a mere guess. Otherwise we would still be at their stage of development. Some beginning had to be made somewhere. There is little doubt that the men of remote antiquity presented many types of make-up, as we do today. Some, for instance, were more curious than others; some more inventive, some more reflective. , which, as we know today, is found even among animals. These more gifted individuals were, as usual, the inventors, discoverers, and builders of systems and language of their period. They could not long fail to recognize the fact that a stone *and* a stone, or a fruit *and* a fruit are *different* from *one* stone or *one* fruit. For instance, the two stones might have saved the early observer's life in defence, or the two fruits might have satisfied his hunger or thirst, where one would not have done so. An accumulation of objects was obviously somehow different from a single object. As these problems were often of vital importance to their lives, names for such accumulations of objects began to be invented, and one and one was called two, two and one was called three, . Number and mathematics were born as a structural semantic life-necessity for a time-binding class of life. They were an expression of the neurological structure and function and of the tendency toward induction.

In the beginning, names and generalizations were made from the simplest brute facts of life, and our primitive ancestors did not realize, that these crude generalizations might not have a structural validity, which they seldom doubted that they possessed, even as we today, seldom doubt. Those primitive scientists, (and we today differ very little from them), having produced *terms*, objectified them, and began to speculate about them. Let us examine some examples of such primitive mathematical speculations. Addition, of course, by which we *generate numbers,*— one and one make two, two and one make three. , was all-important. They could not miss the simple fact that three, which is equal to two and one, by definition, is more than two or one. A primitive generalization; namely, that the sum is always more than the summands taken separately, was still further generalized to a postulate that a part is smaller than the whole. This generalization has hampered mathematics almost up to our own day, and for many thousands of years it prevented the discovery of the notion of mathematical infinity, which we have already discussed in Chapter XIV.

It must be noticed that such generalizations involve *s.r*, which are objective and un-speakable. If verbally formulated they should have a structure similar to that of the facts, otherwise they are fanciful and vicious, because not properly formulated. When formulated they become public structural facts (*s.r* are personal, individual, non-transmittable, and un-speakable) and so they may be criticized, improved; revised, rejected, . All human history shows that the correct structural formulation of a problem is usually as good as the solution of it, because sooner or later a solution always follows a formulation.

After many thousands of years—in fact, practically only the other day—it was found that these primitive generalizations were in general not valid. Negative numbers were invented, and two plus minus-one was no more three but one, 2 + (-1) = 1. The sum was no longer greater than its summands. The usual tragedy takes place here also. A few people know the facts, but the old primitive structural *s.r* *remain* in some of these few, as well as in the great majority of us who did not even know the facts. That such structural *s.r* do not vanish quickly, or generally, is proven again and again throughout history. We see it very clearly in the problems of 'infinity', or $\overline{E}$ geometries, or $\overline{N}$ physics. But the most pathetic sight is to see scientists who have *rationalized* the technique without a deeper re-education of their *s.r*. This is most clearly seen in the case of many writers on the foundations of mathematics, the Einstein theory or on the newer quantum mechanics. They *feel* in the old structural way, they *rationalize* in the new; hence their works are full of self-contradictions. Readers and students alike feel how 'difficult' and messy the whole subject is. As a matter of fact, the new theories are neither messy nor difficult. They are really much simpler and easier than the old theories, *provided* our structural *s.r* are purged of the primitive structural tendencies to which every one of us is heir. When this semantic re-education of our structural feelings is accomplished it is the old that becomes ‘unthinkable’ and incomprehensible, because it gives such a structural mess.

Something similar might be said about a feeling deep-rooted in all of us namely, the ‘plus’ feeling. In all the advances of science we struggle against it. For instance, the example of the green man-made leaf given previously shows clearly that man-made affairs may with some plausibility be considered as ‘plus’ affairs, but not so with non-man-made natural leaves, which appear not as ‘plus’, but functional affairs, where the greenness was structurally *not added* but happened, or became. As a structural fact, the world around us is *not* a ‘plus’ affair, and requires a functional representation. In chemistry, for instance, does hydrogen ‘plus’ oxygen produce water, H_2O ? If we mix the two gases, two parts of hydrogen with one of oxygen we do not get water. We must first pass a spark through the mixture, when an explosion occurs and the result becomes *water*, a *new* compound quite *different* from its elements or from a mere mixture of them. Does one gallon of water and one gallon of alcohol make two gallons of a mixture ? No, it makes less than two gallons. Does light added to light make more light ? Not always. The phenomena of interference show clearly that light ‘added’ to light sometimes makes darkness. Four atoms of hydrogen, of atomic weight 1.008, produce, under proper conditions, one atom of helium, not of atomic weight 4.032, but of atomic weight 4. The 0.032 has somehow mysteriously vanished. Such examples could be quoted endlessly. They show unmistakably that structurally this world is not a ‘plus’ affair, but that *other* than additive principles must be looked for.

The struggle against this ‘plus’ feeling is quite evident, but often unsuccessful, in scientific literature. Man ‘is’ an animal ‘plus’ something. Life ‘is’ ‘dead matter’, ‘plus’ some ‘vitalizing principle’, . In scientific literature we find curious expressions: as for instance, ‘It is impossible to express the conduct of a whole animal as the algebraic sum of the reflexes of its isolated segments’; or, ‘The individual represents heredity *plus* environment’; or, ‘That the abstraction does not merely take away from a number of engram groups some components and combine the rest into one sum, but forms thereby a new psychic structure is self evident and is in no way peculiar to the psyche. Thus a clock work is as little the mere *sum* of its little wheels as a human being is the *sum* of his cells and molecules’; and later on, ‘to be exact the ego consists of the engrams of all our experiences *plus* the actual psychism’. There is endless material that might be quoted, but for our purpose these few samples will suffice. We do not give them with the purpose of citing authoritative examples of the need of non-plus considerations. Far from it. We do it to emphasize the astounding fact that, although the best men in their fields have vaguely felt this necessity, yet even they become a prey to this very old structural linguistic semantic tendency. In all three cases quoted the authors were of the best we have. They have fought all their lives against the ‘plus’ tendency and methods; and yet, if they succeed in eliminating this tendency from one part of their subject, they plant it quite obviously somewhere else. We see that we are dealing here with an ingrained psycho-logical tendency which can be remedied only by a fundamental, $\overline{A}$, structural, semantic investigation.

Let us analyse these quotations. In the second case, we hear, after a successful attack on *plus* tendencies, a statement that the ‘individual represents heredity *plus* environment’. Is this statement true ? Let us take examples. There are certain fishes which are heliotropic and swim toward the light, but if we change the temperature of the water they become negatively heliotropic and swim *away* from the light. Is this most complex activity of the organism-as-a-whole a ‘plus environment’ fact, or does the change of temperature produce some fundamental functional changes ? When, for instance, a good mother rat, having been put on a different though still abundant diet, which is deprived of some minute amount of special vitamins, begins to eat her litters, is this again a ‘plus’ reaction, or is it a most complex functional change of the organism-as-a-whole ? Or when a human being, because he received in childhood an ‘emotional’ shock through outside events (action or language of parents, for instance) develops a functional disorder, or even a physical ailment, is this again a ‘plus environment’ problem ? Or, when chickens fed on eggs laid by hens kept without sunlight or violet rays, or which have only received sunlight through a glass window, develop rickets and soon die, though they do not do so when the glass windows are removed and the sunlight is allowed to operate directly upon the hens. Is this again a ‘plus environment’ example ?

One ‘Smith’ and one ‘Smith’ make two ‘Smiths’, as far as theatre or railway tickets are concerned, but in life, under proper conditions, they form a family and very often many more than two ‘Smiths’ come out of such ‘addition’. How about their work ? Is it a mere sum ? In the case of inventors who may have been influenced by one or many men directly or indirectly, do their inventions produce a sum of the work of as many men ? Surely the steam engine or the dynamo produces more work than not only the inventors, but the series of other men who have been indirectly responsible for the inspiration of the inventors, could ever have produced. So again it is not a ‘plus’ affair.

In the third case we see the author attacking the ‘plus’ tendency on one page, and planting another ‘plus’ a few pages further on, which implies at once some objectified *additional* entity. In this respect it must be noticed that this *additive* tendency represents a partial and important structural and semantic *mechanism of identification,* and to deal successfully with it, we must clear up the problem connected with the *additive* tendency.

The numberless and endless ‘philosophical’ volumes, for instance, which have been written about the ‘body-soul’ problems, show the tremendous structural and semantic importance of the clearing up of this ‘plus’ versus ‘non-plus’ issue. The reader may recall that the $\overline{A}$, the $\overline{E}$, and the $\overline{N}$ systems have one underlying structural metaphysics. The $\overline{E}$ systems deal with non-linear equations and with curved lines, of which the linear equation and the straight line, (one of zero curvature), are only particular cases. And the general theory of Einstein, which is the foundation of $\overline{N}$ systems, also introduces non-linear equations. Ought we to be surprised to find that a $\overline{A}$-system must also solve this difficult structural and semantic problem of linearity versus non-linearity, of additivity versus non-additivity ?

Indeed the problems demanding our attention are extremely baffling and difficult. Even in such a perfected science as physics, we have great difficulties in using non-linear equations, and are still at the stage where we solve few equations other than linear ones. To make any progress at all we must start with the *simplest* available problems in this field; namely, mathematical problems. The main point at this stage is not a solution of the problem but its formulation. When formulated and brought to the attention of mankind, there is no doubt that it will be eventually solved.

To better understand the additive principle, let us consider a group of elements, the individuals of which we denote by letters *a, b, c, d,* . Let us take two or more of these elements and produce a synthesis which results in a third or *n*-th entity. Let this synthesis be of such a nature that the characteristics ascribed to the elements are also present in the resultant synthesis, in other words, let them have the so-called group characteristic. If our elements are, for instance, numbers, the new synthesis is also a number and belongs to the original group. We must notice that the problem of *order* is important in the formulation of the additive principle. If *a* and *b* are the two elements the synthesis of which we define, we must be clear that *a* first and *b* second, or *b* first and *a* second, must be recognized in the synthesis. Let us assume also that only the two alternative orders *a* and *b*, or *b* and *a*, are of importance in this case. The commutative law asserts that *a* plus *b* is equal to *b* plus *a*, $a + b = b + a$, which means that the two possible alternative orders give equivalent results. We must notice that this does not mean that order does not enter into this synthesis; in such a case the above mentioned commutative law would make no assertion at all. It is of importance that order should be involved in the synthesis. It is a matter of indifference only as far as equivalence by a commutative law is concerned.

We should notice for our purpose that the synthesis has the ‘same’ characteristics as the elements had. In other words, if we know the characteristics of the elements we know the characteristics of the result. For instance, if the elements were numbers, the result will be a number, and no characteristic absent in the elements will appear in the result. This predictability from the characteristics of the elements to those of the result is perhaps one of the most striking characteristics of additivity. On the one hand, it allows us to foretell the future; on the other hand, it limits considerably the applicability of the additive principle. It is obvious that when we combine elements, and the results have *new* characteristics absent in the original elements, the new problems are structurally no more of an additive character, and the synthesis must be different.

Only a few of the simplest entities in physics possess additive characteristics. If we take, for instance, ‘weight’ or ‘length’ or ‘time’, we see that these units are additive. One pound, or inch, or second, if added respectively to one pound, or inch, or second, gives us two pounds, or two inches, or two seconds.

Not so, however, with temperature, or density, or many other derived magnitudes, as we call them. If we have a body of temperature of one degree and combine it with another body of equal temperature the synthesis will not have a temperature of two degrees, (as in the case of weight), but of one degree. This applies to density. , two bodies of density one each will not give us a body of density two, but of density one.

Before further analysis of the problems of linearity and additivity, it will be well to consider a few definitions.

If an entity u is changed into an entity v by some process, the change may be regarded as the result of an operation performed upon u, the operand, which has converted it into v. If we denote the operation by f, then the result might be written as $v = fu$. The symbol of the operation f is called the *operator*. We are familiar with many such; indeed the symbols for all mathematical operations may be treated as operators. So for instance the symbol $\sqrt{}$ indicates the operation of extracting the square root. If we deal with a range of values for a variable x, what we have defined as the function symbol $f(x)$ may be treated as an operator whose operation on x may be indicated by the symbol fx. The operation of differentiation may be symbolized by D, the result of whose operation on the variable u, Du is the derivative of u. The sign of the definite integral integral $\int_b^a$ may be taken as indicating an operation which converts a function into a number, .

It is important to know that many of the rules of algebra and arithmetic when defined in this way, give rise to a calculus of operations. The fundamental notion in such a calculus is that of a product. If u is operated upon by f the result v is indicated by fu, or symbolically, $v = fu$. If v in turn is operated upon by g the result w is indicated by gv, or symbolically, $w=gv=gfu$, whence the operation gf which converts u directly into w is called the product of f and g. If this operation is repeated several times in succession the usual notation of powers is used, for instance $ff=f^2$, $fff=f^3$, . Not applying the operator at all, which we would denote by f^0, leaves u unchanged, which we indicate symbolically by the equation $f^0u=u$. The operator f^0 is equivalent to multiplication by 1, $f^0 = 1$, whence f^0 may be called the *idem operator*. We see also that the law of indices holds; namely, the $f^m x f^n = f^{m+n}$.

For our purpose we will analyse only one special case; namely, where we have u, v and $u+v$ as operands, and such an operator, f, that $f(u+v) = fu+fv$. Expressed in words, this means that the operator applied to the sum of the two operands gives a result equal to the sum of the results of operating upon each operand separately. Such a special operator is called a *linear*, or distributive, operator.

In terms of functions we would have $f(x+y) = f(x) + f(y)$ which may be called a functional equation. It has been proved that such a functional equation has one type of solutions; namely, when f is equivalent to a multiplication by a *constant*, or $fx = cx$. This fact is of great importance for us. Many problems in science are stated in terms of variation. For purposes of analysis a statement that ‘x varies as y’ is written $y = kx$, where k is called a factor of proportionality, which enables us to convert a statement of variation into an equation. If y varies inversely as x, we write $y=k(1/x)$ or $y=k/x$. A multiplication by a *constant* thus introduces a relation of proportionality, hence the importance of proportionality in a world where constants are present.

It must also be noticed that the two fundamental operations of the calculus are *linear* without being equivalent to multiplication by a constant. These are: ‘the derivative of the sum is the sum of the derivatives’, that is $D(u+v)=Du+Dv$; and ‘the integral of the sum is the sum of the integrals’, that is $\int(u+v)dx=\int udx+\int vdx$. But as the fundamental notion of the calculus is to substitute for a given function a *linear* function, in other words, to deal with curves as the limits of vanishingly small straight lines, this linearity underlies structurally all fundamental assumptions of the calculus, and one might say with Weyl that ‘one here uses the exceedingly fruitful mathematical device of making a problem *linear* by reverting to infinitely small quantities’.[1]

A vector is defined roughly as a line-segment which has a definite direction and magnitude, and any quantity which can be represented by such a segment is defined as a vector quantity.

The addition of vectors is defined by the law of the parallelogram, as in the case of two forces. It should be noticed that because of this definition the sum of two vectors *differs* in general from the arithmetical sum of the lengths, and only collinear, or parallel vectors obey the arithmetical summation law.

The introduction by definition of mathematical entities which obey different laws from the usual arithmetical laws is an important structural and methodological innovation. It gives us the useful *precedent* of defining our *operations* to suit our needs. The vector calculus accepted as the definition of the sum of two-vectors the law established *experimentally* in physics for the sum of two forces; and so the vector calculus from the beginning was structurally a particularly useful language in physics. Only since Einstein has the value and importance of the vector calculus for physics become generally appreciated.

If we have two vectors, **a** and **b**, starting from a common origin O and complete the parallelogram as in Fig. 1, then the diagonal of the parallelogram will be the required sum, **a**+**b**, by *definition.*

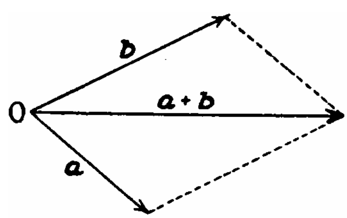

FIG. 1

If we choose two co-initial vectors of unit length, one on the X axis, and the other on the Y axis, and call them **i** and **j**, we can always represent any vector **x** as the sum of two vectors, one of which is the projection of **x** on the X axis, and the other the projection of **x** on the Y axis. (See Fig. 2.) Let us call these vectors **x'** and **x"** respectively. Then **x**=**x'**+**x"**, by definition. But **x'** differs from **i** in length only, hence it can be obtained by multiplying **i** by an appropriate number, say a. Similarly, **x"** can be obtained from **j** by multiplying **j** by b, and so, in symbols, $\mathbf{x}'=a\mathbf{i}$, $\mathbf{x}''=b\mathbf{j}$, and $\mathbf{x}=a\mathbf{i}+b\mathbf{j}$. All vectors of the plane can be obtained from **i** and **j** in this form. The numbers a and b are called com*ponents* of **x**.

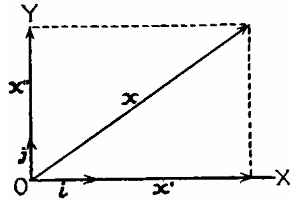

FIG. 2

Now that we know how to express a vector in terms of its components; namely, $\mathbf{x}=a\mathbf{i}+b\mathbf{j}$, let us consider a vector function $f(\mathbf{x})$ which satisfies the equation $f(\mathbf{x}+\mathbf{y})=f(\mathbf{x})+f(\mathbf{y})$. We may take $a\mathbf{i}=\mathbf{x}$ and $b\mathbf{j}=y$ and $\mathbf{x}+\mathbf{y}=\mathbf{z}$ then we have $f(\mathbf{z})=f(\mathbf{x}+\mathbf{y})=f(a\mathbf{i})+f(b\mathbf{j})$. But since a and b are numbers, we have $f(a\mathbf{i})=af(\mathbf{i})$, and likewise, $f(b\mathbf{j})=bf(\mathbf{j})$; so that $f(\mathbf{z})=af(\mathbf{i})+bf(\mathbf{j})$. But $f(\mathbf{i})$ is itself a vector and therefore expressible in the form $a'\mathbf{i}+b'\mathbf{j}$,and,$f(\mathbf{j})=c\mathbf{i}+d\mathbf{j}$.Hence, $f(\mathbf{z}) = a(a'\mathbf{i}+b'\mathbf{j}) + b(c\mathbf{i}+d\mathbf{j}) = (aa'+bc)\mathbf{i} + (ab'+bd)\mathbf{j}$. In general, the components are the coefficients accompanying **i** and **j**, and so we have the components of $f(\mathbf{z}) =f(\mathbf{x}+\mathbf{y})$ in terms of the components of **z**; and we see how the components of a vector are changed into the components of the linear vector function of the vector.

In general terms, a continuous vector function of a vector is said to be a *linear* vector function when the function of the sum of any two vectors is the sum of the functions of those vectors; that is, the function f is linear if $f(\mathbf{r}_1+\mathbf{r}_2)=f(\mathbf{r}_1)+f(\mathbf{r}_2)$, whence, if a be any positive or negative *number* and if f be a *linear* function then the function of a times **r** is a times the function of **r**; $f(a\mathbf{r}) =af(\mathbf{r})$.

Linear vector operators are also defined by a similar equation; namely, $L(\mathbf{a}+\mathbf{b}) = L\mathbf{a} +L\mathbf{b}$.

Let us recapitulate. If we take the functional equation$f(x+y)=f(x)+f(y)$, which might be used as a definition of *linearity*, and which is based on *additivity*, and take $x=y=1$; then we have $f(1+1)=f(2)$ and also $f(1)+f(1)=2f(1)$; and so our original equation becomes by substitution $f(2)=2f(1)$. It is obvious that the original equation, $f(x+y)=f(x)+f(y)$, is the source of indefinitely many such relations for particular numbers. For instance, $f(3)=f(2+1)=f(2)+f(1)$; but, in accordance with what we obtained before, $f(2)=2f(1)$; so that $f(3)=2f(1)+f(l)=3f(1)$, and in general, $f(x)=xf(1)$. So, if we have an equation $f(x+y)=f(x)+f(y)$ for numbers, we know that we can obtain the value of this function for any x if we know it for 1. If we denote the function of 1, which is a *constant*, by $f(1)=k$, we have the general form of the function which satisfied $f(x+yi=f(xi+f(y)$ expressed by $f(x)=kx$. In words, a functional equation of the above type; namely, a function of the sum equal to the sum of the functions, has only one possible type of solution; namely, when f is equivalent to a multiplication by a constant, or, $f(x) =kx$. But this last means proportionality. The values of the function are proportional to the arguments (variables). In fact, let us consider two arguments, that is, two values of the independent variables x and y. We have, as shown before, $f(x)=kx$ and *f*(*y*)=*ky*. Dividing the first by the second, we obtain *f*(*x*)/*f*(*y*)=*kx*/*ky*=*x*/*y* or, in another form, *f*(*x*)/*x*=*f*(*y*)/*y*=*k*.

Let us consider an example. We know from elementary geometry that if we take an angle α and draw parallels which cut the sides at *AA'*, *BB'*, *CC'*, *DD'*. , the corresponding intercepts are proportional. In general, the lengths of the segments on the left side are not equal, neither are they equal to the segments on the right side. If we designate the segment *AB* as *x* and *BC* as *y*, the corresponding segments *A'B'* and *B'C'* we may designate as *f*(*x*) and *f*(*y*), respectively, which means function of *x* and function of *y*. *A'B'=f(x)'*, *B'C'=f(y)*. But the above intercepts are proportional, which means that $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AB+BC}{A'B'+B'C'}$. We easily see from Fig. 3, that *AB*+*BC*=*AC*=*x*+*y;* and *A'B'*+*B'C'*=*A'C'* and so *A'C'* on the one hand is *f*(*AC*)=*f*(*x*+*y*) and on the other hand it is *f*(*x*)+*f*(*y*) and therefore *f*(*x*+*y*)=*f*(*x*)+*f*(*y*). We could multiply examples by taking relations between central angles in a circle and arcs of its circumference. In fact, any problem of *measure* in *E* geometry could be used as an example.

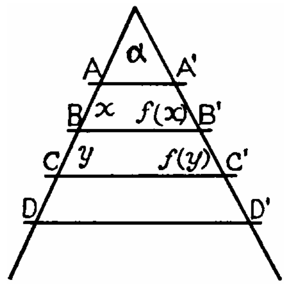

FIG. 3

In our development we started with definite *additive* natural tendencies, not only in our highest, yet undeveloped, mathematics, which we call our daily and scientific language, but also in our lowest, but perfected, language which we call mathematics. In this perfected language the notion of *additivity* is connected with *linearity,* and the *methods* of *approximation* are also founded on additivity and linearity.

Yet the world around us in its more fundamental structural aspects is not additive; and for *adjustment* we must find means of passing from additive tendencies and formulations to non-additive tendencies and formulations. Modern mathematics has developed these methods, and modern physics is beginning to apply them. Let us repeat: the importance of linear functions implies the importance of ‘straight’ lines. They are important on two counts: first, because they are simpler than all other curves, so that naturally we want to study them before we study other curves, such as, for instance, circles or the other conic sections in elementary geometry; and secondly, because all curves can be approximated by straight lines. This point is very important, as approximation is the most powerful method we have of handling complicated situations.

There are two methods of approximating a curve in the vicinity of a point. If we are interested in the immediate vicinity of a point we approximate the curve by its tangent, as the tangent approximates the curve in the vicinity of a point better than any other straight line. If we want to decrease the error which we make in this approximation, we have only to decrease the vicinity in which we consider it. If we do not want to restrict ourselves to a small neighbourhood we have to use more complicated methods of approximation. We inscribe into the curve a broken line which consists of segments of straight lines. The beginnings of the study of curves consist in reducing the study of curves to: (1) The study of straight lines connected with the curves' tangents, which is the point of departure of the differential calculus; and (2) The study of the inscribed broken lines, which is the point of departure of the integral calculus.

Now curves represent only the simplest dependences. In other cases we have more complex kinds of functions; for instance, vector functions; but in every case we have *linear* functions, the simplest of their type; and other functions are studied by approximating them in one way or another by linear functions. In using the term 'function', we mean not merely numerical functions but also *operators*, which are to the ordinary functions what ordinary functions are to numbers. A general definition of linearity can be connected with that of proportionality in the following manner. If two variables are proportional, one to another, then to the sum of any values of the first corresponds the sum of the corresponding values of the second.

The simplest part of any field is the consideration of linear, additive questions; linear equations (equations of first degree in algebra), linear differential equations, linear integral equations, linear matrices, linear operators, . But sooner or later we come to the more difficult and more interesting non-linear problems. Perhaps the main importance of the General Theory of Einstein lies in the fact that the equations of physics become *non-linear*. Now, although non-linear equations can be approximated by linear equations, the character of a world determined by non-linear equations must be entirely different from a world determined by linear equations. In a linear world electrons would not repel each other but would travel independently of each other, and there could be no relation between the charges of different electrons. But we know that electrons do repel each other, and attract protons, and that their charges are equal. In physics, if a system can be described by linear differential equations, the causal trains started by different events propagate themselves *without interference,* with simple *addition* of effects.

The properties of systems which can be described by linear differential equations have, as we have already seen, the property of *additivity.* This means that the result of the effects of a number of elements is the sum of the effects separately, and no new effects will appear in the aggregate which were not present in the elements. In such a universe there is 'continuity', fieds are superposable, wave disturbances are additive, 'energy' and 'mass' are indestructible, . In such a universe we can have two-valued *causality,* as causal trains started by different events propagate themselves without *interference,* and with simple addition of effects, and the present can be analysed backwards into the sum of elementary events, that is, a two-valued causal analysis is possible.

If our equations are not linear, the effects are not additive and a two-valued causal analysis is not possible.

The joint effect of *two* causes working together is not the *sum* of their effects separately[2], and we need ∞-valued causality.

Analytically, if we have *linear* differential equations and we have one solution $y_1=f(x)$ and another solution $y_2=F(x)$ then their sum is also a solution; namely, $y_3=f(x)+F(x)$. If the differential equations are non-linear and if $y_1=f(x)$ and $y_2=F(x)$ are two solutions, then $f(x)+F(x)$ is *not* a solution.

Linear problems and linear equations play a very important structural role in science and there is little doubt that linear equations preponderate enormously, although many fundamental events cannot be described by such equations. A universe which can be described by linear differential equations of the second order has definite structural characteristics—in the main in rough accord with observation. As such differential equations give us the tendency of a process, we may use them to describe large-scale phenomena by integration, or the statistical phenomena of great numbers.

Unfortunately, the study of non-linear problems is structurally very difficult and largely a problem of the future.

There is one very important point which we should not miss. We know already that there is a fundamental difference between different orders of abstractions. Physical abstractions have always characteristics left out, and our higher order abstractions are further removed from life, but they have all characteristics included. The problem of sanity being a problem of adjustment, we must somehow correlate these abstractions in which characteristics are *left out* with those which include all characteristics, and so *must* proceed *by approximations.* Mathematical methods, particularly those of the differential and integral calculus, have evolved the best technique of *approximation* in existence today, which, as we have seen, is strictly connected with *linearity* or *additivity.*

A similar urge which prompted us in the expression of our additive tendencies and methods in the structure of language, has led to the production of the calculus. For organisms which abstract in so numerous and such different orders, the methods of the calculus are therefore fundamental psycho-logical devices, conditioning sanity.

In conclusion, we should notice two quite important facts. One of these is that the nervous system, being in a state of nervous tension, cannot structurally be a simple additive affair in all its functions, a fact which every one of us has experienced. Too many stimulations dull, or abolish, or change reaction in an enormous variety of ways. Piéron, as a result of experimenting in association, has not only shown the complexity of these processes, but also reaches the conclusion that the associative connections are non-linear.[3] The other most important point is that structurally the term ‘and’ implies addition. When we confuse orders of abstractions or levels of analysis, the ‘and’ additive implications falsify the issues. Thus for instance two atoms of hydrogen *and* one atom of oxygen *and* a spark produce water. The second ‘and’, at least, is used illegitimately, as it applies to an entirely different level (the spark) from that of the atoms. Linguistically we introduced additive implications, while empirically we are dealing with most complex non-additive, non-linear higher-degree functions. When we confuse orders of abstractions, as we all do, the ‘and’ is bound to introduce structurally false implications, which it is very difficult to avoid—the more so since these semantic problems are generally entirely neglected.