BOOK III

ADDITIONAL STRUCTURAL DATA ABOUT LANGUAGES AND THE EMPIRICAL WORLD

The every-day language reeks with philosophies . . . It shatters at every touch of advancing knowledge. At its heart lies paradox.

The language of mathematics, on the contrary, stands and grows in firmness. It gives service to men beyond all other language. (25) ARTHUR F. BENTLEY

Nothing is more interesting to the true theorist than a fact which directly contradicts a theory generally accepted up to that time, for this is his particular work. (415) M. PLANCK

It is not surprising that our language should be incapable of describing the processes occurring within the atoms, for, as has been remarked, it was invented to describe the experiences of daily life, and these consist only of processes involving exceedingly large numbers of atoms. Furthermore, it is very difficult to modify our language so that it will be able to describe these atomic processes, for words can only describe things of which we can form mental pictures, and this ability, too, is a result of daily experience. (215) W. HEISENBERG

PREFATORY REMARKS

In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi.
(74) GEORG CANTOR

We cannot describe substance; we can only give a name to it. Any attempt to do more than give a name leads at once to an attribution of structure. But structure can be described to some extent; and when reduced to ultimate terms it appears to resolve itself into a complex of relations . . . A law of nature resolves itself into a constant relation, . . . , of the two world-conditions to which the different classes of observed quantities forming the two sides of the equation are traceable. Such a constant relation independent of measure-code is only to be expressed by a tensor equation. (148) A. S. EDDINGTON

We have found reason to believe that this creative action of the mind follows closely the mathematical process of Hamiltonian differentiation of an invariant. (148) A. S. EDDINGTON

The only justification for our concepts and system of concepts is that they serve to represent the complex of our experiences; beyond this they have no legitimacy. I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in removing certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of the *a priori.*
(152) A. EINSTEIN

In writing the following *semantic* survey of a rather wide field of mathematics and physics, I was confronted with a difficult task of selecting source-books. *Any* mathematical treatise involves conscious and many unconscious notions concerning ‘infinity’, the nature of numbers, mathematics, ‘proof’, ‘rigour’. , which underlie the definitions of further fundamental terms, such as ‘continuity’, ‘limits’, . It seems that when we discover a universally *constant empirical relation*, such as ‘non-identity’, and apply it, then all other assumptions have to be revised, from this new point of view, irrespective of what startling consequences may follow.

At present, neither the laymen nor the majority of scientists realize that human mathematical behaviour has many aspects which should never be identified. Thus, (1) to be somehow aware that ‘one and one combine in some way into two’, is a notion which is common even among children, ‘mentally’ deficients, and most primitive peoples. (2) The mathematical ‘1+1=2’ already represents a very advanced stage (in theory, and in method. ,) of development, although in *practice* both of these *s.r* may lead to one result. It should be noticed that the above (1) represents an individual *s.r*, as it is not a general formulation; and (2) represents and involves a generalized *s.r*. Does that exhaust the problem of ‘1+1=2’ ? It does not seem to. Thus, (3), in the *Principia Mathematica* of Whitehead and Russell which deals with the meanings and foundations of mathematics, written in a special shorthand, abbreviating statements perhaps tenfold, it takes more than 350 large ‘shorthand’ pages to arrive at the notion of ‘number one’.

It becomes obvious that we should not identify the manipulation of mathematical symbols with the semantic aspects of mathematics. History and investigations show that both aspects are necessary and important, although of the two, the semantic discoveries are strictly connected with the revolutionary advances in science, and have invariably marked a new period of human development. In Chapter XXXIX, the reader will find a very impressive example of this *general* fact. Thus, what is known as the 'Lorentz transformation', *looks like* the 'Einstein transformation'. When manipulated numerically both give equal numerical results, yet the meanings, and the semantic aspects, are different. Although Lorentz produced the 'Lorentz transformation' he did not, and *could not* have produced the revolutionary Einstein theory.

It is well known that when it comes to the manipulation of symbols mathematicians agree, but when it comes to the semantic aspects or meanings. , they are *admittedly* hopelessly at variance. In a prevailingly *A* world we have had no satisfactory theory of 'infinity', or a $\bar{A}$ definition of numbers and mathematics. This necessarily resulted in the fact that the semantic aspects of practically all important mathematical works by different authors often involve *individual semantic presuppositions*, or orientations concerning fundamentals. My presentation intends to be primarily semantic and elementary, and is only remotely concerned with the manipulation of symbols. A $\bar{A}$-system, which rejects 'identity', differs very widely from *A* attitudes, and introduces distinct $\bar{A}$ requirements. I had, therefore, to select from many works, with their *individual* presuppositions, those which were less in conflict with $\bar{A}$ principles than the others.

A survey of important mathematical treatises shows that although the majority of modern mathematicians explicitly abjure the 'infinitesimal', yet, in some presentations, this notion persists. In my presentation I reject the 'infinitesimal' explicitly and implicitly, although the formulae are not altered. 'Modern' calculus is based *officially* on the theory of limits, but as the theory of limits involves the unclarified theory of 'infinity'. , nothing would be gained semantically and for my purpose, had I stressed these formal possibilities of the calculus. Quite the opposite, if I had done so, I would have failed to stress the most fundamental $\bar{A}$ principle and task of establishing the *similarity of structure between languages and the un-speakable levels and happenings as the first and crucial consequence of the elimination of identity*. For these weighty reasons, in my presentation, I followed some older textbooks, particularly Osgood's, which, from a $\bar{A}$ point of view, are sounder than the newer, largely *A* rationalizations and apologetics.

However, it should be realized that practically all outstanding and creative mathematicians have had, and still have, $\bar{A}$ attitudes, yet, these private beneficial attitudes, not being formulated in a $\bar{A}$-system, could not become *conscious*, simple, workable, public, and educational assets. We can be simple about this point. With the elimination of identity, structure becomes the *only possible* content of 'knowledge'—and structure of the un-speakable levels has to be *discovered*. Discovery depends on the finding of *new*, and therefore *different* characteristics. In the *formulation* of the last sentence, we cannot make the 'training in discovery' an *educational discipline.* The opposite is true in a $\bar{A}$-system, based on non-identity, as *we can train simply and effectively in non-identity,* which ultimately leads to differentiation, and so discovery.

Because of the elementary, and purely semantic character of the following pages, I have often restrained myself from giving technical, supposedly 'rigorous', and often *A* rationalizations, which we occasionally call definitions. In a semantic and $\bar{A}$ treatment, at this pioneering stage, stressing old definitions would be seriously confusing; and I wished to avoid such witty wittgensteinian 'definitions' as 'A point in space is a place for an argument'. In a number of instances, and for my purpose, I often avoided unsatisfactory formal definitions, preferring to depend upon the ordinary meanings of words.

For the reader who wishes to acquaint himself with an elementary theory of limits and corresponding sets of definitions, I would suggest the book of the late Professor J. G. Leathem, *Elements of the Mathematical Theory of Limits* (London and Chicago, 1925). This theory is based on Pascal's *Calcolo Infinitesimale,* Borel's *Théorie des fonctions,* and Godefroy's *Théorie des séries.* Leathem's book has been printed under the supervision of Professor H. F. Baker, F.R.S., of the University of Cambridge, and Professor E. T. Whittaker, F.R.S., of Edinburgh. I give these names for professional mathematicians, to indicate the semantic trend which underlies this particular treatment of limits and which does not greatly conflict with a $\bar{A}$ outlook. This outlook may be summarized in part, in the words of Borel somewhat as follows: 'To the evolution of physics should correspond an evolution of mathematics, which, without abandoning the classical and well-tried theories, should develop however, with the results of experiments in view'. This statement implies vaguely the 'similarity of structure'. , and so requires as a *modus operandi* the rejection of identity.

There seems to be little doubt that a complete and radical revision of the semantic aspects of human mathematical behaviour is pending. Such a revision appears to be laborious and difficult, and should be undertaken from the point of view of the theory of the unique and specific relations, called numbers. I doubt if a single man could accomplish this revision. Such an undertaking will probably be the result of group activities, and may, in the beginning, be *unified* by the formulation of one fundamental $\bar{A}$ principle of non-identity, the disregard of which, from science down to 'mental' ills, can be found at the bottom of practically all avoidable human difficulties.

The problems are very complicated and extremely difficult, and need to be treated from many angles. At present, we have many scientific societies, grouped by their specialties; but we do not have a scientific society composed of *many different specialists* whose work could be unified by some *common and general principle.* There can be no doubt that the principle of 'identity', or 'absolute sameness in *all* aspects', is invariably false to facts. The main problem is *to trace* this semantic disturbance of improper evaluation in all fields of science and life, and this requires a new *co-ordinating scientific body* of many specialists, with branches in all universities. Each group would meet, say monthly, to discuss their problems, and give mutual technical assistance in tracing *this first general semantic disturbance.* Such meetings would stimulate enormously scientific productivity. In fact, without such a co-ordinating body, the present enormous technical developments in each branch of science preclude the revision of general principles, on which, in the last analysis, all other of our activities greatly depend. The first task then, is to find a co-ordinating principle, and present it to the scientific world.

Psychiatry, and common experience, teach us, that in heavy cases of dementia praecox we find the most highly developed 'identification'. $\overline{A}$ considerations suggest that *any* identification, no matter how slight, represents a dementia praecox factor in our semantic reactions. The rest is only a question of degrees of this maladjustment. From this point of view, we will find dementia praecox factors even in mathematics. In physics, only since Einstein has this factor of un-sanity been eliminated, and this elimination has already produced an ever-growing crop of 'geniuses', which merely means, that some inhibitions of mis-evaluation have been eliminated from these younger men, and that they are humanly more 'normal' than the others.

In mathematics, from a $\overline{A}$ point of view, we must first of all *not identify* different aspects of our mathematical behaviour, nor try to cover up these identifications of endless aspects by the one very old term 'mathematics'. This word, 'mathematics', in its *accepted sense* covers a non-existing fiction. What does exist, and the only thing we actually deal with, is *human mathematical behaviour,* human *s.r,* and the *results* of human *mathematical behaviour* and *s.r.* A treatise, say, on a new quantum mechanics, has no value to a monkey or a corpse, and only human *mathematical behaviour* and *s.r,* have any actual *non-el* existence, and is the *only* thing which actually matters. So we see that 'mathematics' covers a non-existent fiction *if elementalistically* separated from human mathematical behaviour and *s.r.* I use the term 'mathematics' in the *non-el* sense, and attempt to signalize some of the difficulties non-elementalism involves at this transitory stage.

From a $\overline{A}$, non-identity, structural, *non-el* point of view, human mathematical behaviour must be treated uniquely as a physico-mathematical discipline, and postulational methods to be used exclusively as a most valuable *checking* method. To *base* mathematical behaviour and *s.r* on postulational methods exclusively, is to introduce dementia praecox factors into science, which only induces the spread of semantic maladjustment in life.

Our main task in producing a $\overline{A}$ revision of mathematical *s.r,* is in the elimination of identification from our *s.r* about 'infinity' and in the formulation of a $\overline{A}$ definition of numbers in terms of relations. This would enable us to rebuild human mathematical *s.r* from a theory of numbers point of view, as a *physico-mathematical* discipline. The intrinsic, or internal theory of surfaces, and the tensor, or absolute calculus, are methodologically our most secure epistemological guides.

I would suggest that mathematical and scientific readers who are interested in a $\overline{A}$ revision should, at first, in their special fields, sketch in technical papers, presented before the local International Non-aristotelian Societies, *A* pitfalls and $\overline{A}$ problems and outlooks. Only after this is done, shall we be able to begin a co-ordination of their findings, and thereby initiate a revised and unified $\overline{A}$ science, mathematics, and perhaps ultimately a saner *scientific civilization.*

The scientific achievements dealt with in Book III, are developing so rapidly, and the technical points of view alter so often, that on a static printed page it is impossible to do them justice. The writer has spared no efforts to keep informed of these scientific developments until two weeks before the appearance of this book; yet because these new developments do not represent new and fundamental semantic factors, I deliberately do not include them here. In some instances, a given author may seem to change his opinions, but, from a $\overline{A}$ point of view, it sometimes appears that the original notions were more justified, and so I preserved them without change.

The following pages are written exclusively from a semantic point of view, an undertaking which is far more difficult than dealing with a restricted technical physico-mathematical problem, because it involves *second order* observations, of the first order observations, of the first order observer, and of the relations between them, . When it came to a final revision of the manuscript, and reading of the proofs, I found that dealing with so many varied fields, languages, and symbolisms at one period, was no small task, and I only hope that I have not over-looked too many errors or misprints.

If we must have slogans, a $\overline{A}$ motto readily suggests itself—'Scientists of the world unite'. Perhaps this motto may prove more constructive and workable than the familiar *A elementalistic* slogans which have mostly led to the dismembering of human society. Protests against any misrule should not be confused with the proclaiming of disrupting *general principles.* Let me repeat once more, that the most lowly manual worker is useful *only* because of his human nervous system, which produced all science, and which differentiates him from an animal, and not primarily for his hands alone; otherwise we would breed apes to do the world's work.

In the explanations of some geometrical notions, and some parts of the theory of Einstein, I have followed often very closely the *Einstein's Theory of Relativity* by Max Born, which is easily the best elementary exposition I have read, and also the books of Eddington. In the quantum field I have followed mostly the books by Biggs, Birtwistle, Bôcher, Haas, and Sommerfeld, and I wish to acknowledge my indebtedness to the above authors.

I am also under heavy obligations to Professors E. T. Bell, P. W. Bridgman, B. F. Dostal, R. J. Kennedy, and G. Y. Rainich, who were so kind as to read the MS. and/or proofs, and whose criticism and suggestions were invaluable to me. However, I assume entire responsibility for the following pages, especially since I have not always followed the suggestions made.

PART VIII

ON THE STRUCTURE OF MATHEMATICS

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can. (510) SILVANUS P. THOMPSON

Besides the theory of surfaces is the model on which all the higher theories are built and must be built, and it is well to master it completely before attempting generalizations. (425) G. Y. RAINICH

To find such relations Einstein has applied a mathematical method of great power—the calculus of tensors—with extraordinary success. The calculus threshes out the laws of nature, separating the observer's eccentricities from what is independent of him, with the superb efficiency of a modern harvester. (21) E. T. BELL

CHAPTER XXXII

ON THE SEMANTICS OF THE DIFFERENTIAL CALCULUS

> **The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts** is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry, and, indeed, the mainspring of all the eminent work of Riemann, in particular, that dealing with the theory of complex function. (547) HERMANN WEYL
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> The conception of tensors is possible owing to the circumstance that the transition from one co-ordinate system to another expresses itself as a **linear** transformation in the differentials. One here uses the exceedingly fruitful mathematical device of making a problem "linear" by reverting to infinitely small quantities. (547)HERMANN WEYL

Section A. Introductory.

In the first draft of this book written in 1928, the following pages preceded Part VII. In a final revision in 1932, it seemed advisable to transfer pages which to laymen look 'mathematical', to the end of the volume, because the majority of even intelligent readers have a sort of 'inferiority complex' about anything 'mathematical'.

The patient reader knows by now, I hope, that on neurological grounds, he must for the sake of sanity, be able to translate the dynamic into the static, and the static into the dynamic; and also that he must know at least about the modern structure of 'space', 'time', 'matter'. These conditions seem essential for sanity, and so I had no choice but to give the minimum of a structural and semantic outline, and to acquaint the reader with the existence of modern scientific problems and vocabularies. It is not my aim to teach the reader mathematics or modern physics. I must limit myself to structural and semantic issues, for there are excellent elementary books which will give him the necessary informations.

The following pages should in no way intimidate the intelligent reader. Elementary structural statements and definitions are given in simple language, followed by illustrations to render their meanings more understandable. The pages are less technical than they look, as each example is carried through in the most elementary way in all of its details, so as to make easy reading. A real difficulty for some readers may come from the semantic blockage created by the use of apparently strange, and, to them, unknown terms, or from a feeling of fright or abhorrence of anything mathematical, due to deplorably faulty introduction to some branch of mathematics at the hands of some teacher innocent of the broader epistemological aspects of science. I am acquainted with scientists of very considerable mathematical gifts, who have had to overcome this phobia of mathematics. Once the word 'mathematics' was mentioned to them, they became 'mentally' paralysed. An 'emotional' fright seized them and it took some months to overcome this undesirable childish *s.r.* I use the subject of mathematics as an illustration of this difficulty, because I want to contrast the comparative simplicity of mathematical notions with the complexity of human problems and language. For when we have understood the *simplest* notions, which happen to be mathematical, then only shall we be able better to understand our human problems, which are in comparison so difficult and so confused.

Any reader who has a distaste for mathematics will benefit most if he overcomes his semantic phobia and struggles through these pages, even several times. As a result of so doing he will find it simple although not always easy. It is always semantically useful to overcome one's phobias; it liberates one from unjustified fears, feelings of inferiority, . The main point of this whole discussion is to evoke the semantic components of a living Smith, when he habitually uses the method which will be explained herewith. This method is so simple and so fundamental that in the form given by a $\overline{A}$-system and further simplified according to the gifts of the teacher, it will some day be introduced into *elementary* schools without technicalities, as a *preventive* semantic method against 'insanity', un-sanity and other nervous and semantic difficulties, as a foundation for a training in *sanity* and adjustment.

Section B. On the Differential Calculus.

1. *GENERAL CONSIDERATIONS*

As we have already seen, the structural notion of a function is strictly connected with that of the variable. The variable on one level does not 'vary'; it is a selection by Smith of a definite value from a given set. As these processes are going on inside of the skin of Smith he might experience on a different level a feeling of 'change'. The method of dealing with such problems is given by the mathematical differential and integral calculus.

The beginnings of methods dealing with 'change' are to be found even among the ancients. Galileo, Roberval, Napier, Barrow, and others were interested in 'fluxional' methods, before Newton and Leibnitz.[1] The epoch-making discoveries of the last two mathematicians consisted not only in perfecting the knowledge they had and in inventing new methods, but also—and this is perhaps the most important—they formulated a *general* theory of these structural methods and invented a new notation suitable for their purpose. The definite abandonment of the old tentative methods of integration in favour of methods in which integration is regarded as the inverse of differentiation was especially the work of Newton. Leibnitz' main work was in the field of precise formulation of simple rules for differentiation in special cases and the introduction of a very useful notation.

It is not an exaggeration to say that the calculus is one of the most inspiring, creative, structural methods in mathematics. There is little doubt that the analysis of the foundations of mathematics, and their revision, was suggested by a study of the methods of the calculus. *It is structurally and semantically the 'logic' of sanity* and, as such, can be given ultimately without technicalities by the present $\overline{A}$-system and semantic training, with the aid of the Structural Differential.

The application of the differential calculus to geometry produced differential geometry. This prepared the way for the notions of Einstein and Minkowski.

The whole of modern physics becomes possible through the calculus, and it will probably be correct to say that the achievements of the future also will be dependent on it.

The present work is also to a large extent inspired by it, and develops simple non-technical methods by which the psycho-logical structural *s.r* necessitated by the calculus can be given to the masses in elementary education without any technical knowledge of it. This statement does not include teachers, who should be acquainted with at least the rudiments of the calculus.[2]

It is true that in the beginning we did not suspect that the semantics of the calculus are indispensable in education for *sanity*. It is the *only* structural method which can reconcile the as yet irreconcilable higher and lower order abstractions. Without such a reconciliation, at our present level of development, sanity is a matter of good luck quite beyond our conscious or educational control.

Let us recall the rough definition of a function: y is said to be a function of x if, when x is given, y is determined. In symbols we write $y=f(x)$ which we read 'y is equal to a function of x' or 'y is equal to f of x'. If y is a function of x, or $y=f(x)$, then x is called the independent variable, being the one to which we arbitrarily assign *any* value we choose out of a given set of values. The y is called the dependent variable as its value depends on the value we assign to x.

A function may have more than one independent variable; in which case we have a function of several variables. It happens frequently that to one value of the independent variable there may correspond several values of the dependent variable. Then y is said to be a multiple-valued function of x.

Roughly speaking, a function is said to be continuous if a small increment in the variable gives rise to a small increment of the function.

A theory of functions can be developed without any references to graphs and geometrical notions of co-ordinates and lengths; but in practice (and in this work), it is extremely useful to introduce these geometrical notions, as they help intuition. A modern definition of an analytic function is technical and unnecessary for our purpose. Suffice it to say that it is connected with derivatives and power series, *which means structure.*

Geometry is a very remarkable science. It may be treated as pure mathematics, or it may be treated as physics. It may therefore be used as a link between the two or as a link between the higher and lower order of abstractions. This fact is of tremendous psycho-logical and semantic importance. It is not by pure 'chance' that the most important writers on mathematical philosophy, authors who have generalized their knowledge of mathematics to include human results, were mostly geometers.

Indeed, Whitehead, in his *Universal Algebra* (p. 32), says, and justly so, that a treatise on universal algebra is also a treatise on certain generalized notions of 'space'. 'Space' should be understood as 'fulness', 'fulness of something', a plenum. Naturally coherent speech, like universal algebra, must be coherent speech about *something*. 'Generalized space' becomes generalized plenum, and so it belongs to *two* realms. One is contentless and formal, hence generalized algebra; the other, in that it refers to a generalized plenum, becomes generalized geometry, or generalized physics

The main importance, perhaps, of geometry is in the fact that it can be interpreted *both ways*. One way appears as pure mathematics, and therefore as the study of sets of numbers representing co-ordinates. The other takes the form of an interpretation, in which its terms imply a connection with the empirical entities of our world. Obviously if speech is not the things spoken about, we must have a special discipline which will translate the coherent language of pure mathematics, which is contentless by definition, into another way of speaking which uses a different vocabulary capable of *both* interpretations.

Again, the different orders of abstractions, which our nervous structure produces, are perfectly reflected in the very structure and methods of mathematics. The possibility of the use of the 'intuitions' of lower order abstractions, is extremely useful in pure mathematics. This fact makes geometry also *unique*. It allows us to apply to the development of geometry both orders of abstractions—the 'intuitions', 'feelings', of the lower order of abstractions, and the static, 'quantum' jump methods of pure analysis. This is also why the einsteinian physics becomes four-dimensional geometry; which, because it can be treated on both levels of abstraction, gives tremendously powerful and important psycho-logical means for sanity and nervous co-ordination of the individual. Since Einstein, many far-sighted scientists have said that although they do not know in what respect the Einstein theory will affect our lives, yet they feel that it will have a tremendous influence. I venture to suggest that the bearing of the Einstein theory and its development on the problems of sanity, as explained in this work, is a new and unexpected semantic result of the application of modern science to our lives. As the Einstein theory could have been formulated more than two hundred years ago when the finite velocity of light was discovered, so the present theory is also several hundred years overdue. The only consolation we have left is that it is better late than never.

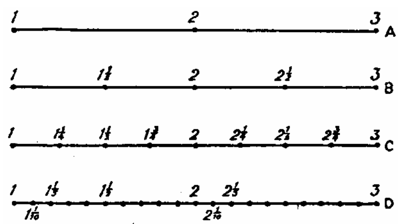
The scope of this work allows us to go but a little beyond these simple remarks, and permits only a very brief explanation of the most fundamental and elementary beginnings of the calculus. In this presentation I shall appeal very often to intuition (lower order abstractions), as this will help the reader.

The notion of differentiation of a continuous function is the process for measuring the rate of growth; that is to say, the evaluation of the increment of the function as compared with the growth or increment of the variable. We may describe this process as follows: If y is a function of x, it is helpful not to consider x as having one or another special value but as flowing or growing, just as we feel 'time' or follow the ripples made by a stone thrown into a pond.

The function y varies with x, sometimes increasing, sometimes decreasing. We have already defined the variable as *any* value selected from a given range. Let us consider our x as given in the interval between 1 and 5. We are now interested in all values which our *x* may take between these two values, or, as we say, in this interval. Obviously, we can select a few values, or, in other words, take big steps; as, for instance, assigning to *x* the successive values x_1=1, x_2=2, x_3=3, x_4=4, x_5=5. In such a case we would have few values and the difference between two successive values would be rather large, for instance, x_3-x_2 = 1. But such large differences are not of much interest to us here. We may, if we choose, select smaller differences; in other words, assign more values to our variable in the given range.

Let us take, for instance, for our *x* the series of values: 1, 1½, 2, 2½, 3, 3½, 4, 4½, 5. Here we see that the difference between two successive values is smaller than 1, it is ½. So we already have nine, instead of the former five, values which we may assign to our *x*. Thus we have selected smaller steps by which to proceed. Let us select still smaller steps; for instance, ¼. Our extensional manifold of values for *x* in the interval between 1 and 5 would then be: 1, 1¼, 1½, 1¾, 2, 2¼, 2½, 2¾, 3, 3¼, 3½, 3¾, 4, 4¼, 4½, 4¾, 5. We see that in the interval between 1 and 5, we have already 17 values which we may assign to our variable, but we have followed the 'growth' of our *x* by smaller steps; namely, by steps of ¼. If we choose to diminish the steps to 1/10, we would have for our extensional manifold of values: 1, 1.1, . . . , 1.9, 2, 2.1, . . . , 2.9, 3, 3.1, . . . , 3.9, 4, 4.1, . . . , 4.9, 5: in all, 41 values for *x*, any two succeeding values differing by 1/10. If we select still smaller steps—let us say, 1/100—we have 401 values for *x* and the difference between two successive values is still smaller; namely, 1/100. This process may be carried on until we have as many numbers between 1 and 5 as we choose, since we may make the difference between successive numbers in the sequence as small as we please. In the limit, between any two numbers, let us say. 1 and 2, or any two fractions, there are infinite numbers of other numbers or fractions. It is obvious that in a given interval, let us say, between 1 and 5, we can have an indefinitely large number of intermediary numbers arranged in an increasing progression, such that the difference between two successive numbers can be made smaller than any assigned value, which is itself greater than zero.

The above may be made clearer by a geometrical illustration. Let us take a segment of a line of definite length, let us say 2 inches. Let us designate the ends by numbers 1 and 3. In figure (A) we divide the segment into 2 equal parts of one inch each, and see that to reach 3 starting with 1 we have to proceed by two large jumps from 1 to 2, and from 2 to 3. In figure (B) we have more steps in the interval and therefore the steps are smaller. In figures (C) and (D) the steps are still smaller and their number greater. If the number of steps is very large, the steps are very small. In the limit, if the numbers of steps become infinite, the length of the steps tends



toward zero and the aggregate of such points of division represents (in the rough only) a *continuous* line.

It is important that the reader should become thoroughly acquainted with the above simple considerations as they will be very useful in *any* line of endeavour. Here we already have learned how, somehow, to translate discontinuous jumps into ‘continuous’ smooth entities. Because of the structure of our nervous system we ‘feel’ ‘continuity’, yet we can analyse it into a smaller or larger number of definite jumps, according to our needs. The secret of this process lies in assigning an increasing number of jumps, which as they become vanishingly small, or tend to zero, as we say, cease to be felt as jumps and are felt as a ‘continuous’ motion, or change, or growth or anything of this sort.

An excellent example is given by the motion pictures. When we look at them we see a very good representation of life with all its continuity of transitions between joy and sorrow. If we look at an arrested film we find a definite number of *static* pictures, each differing from the next by a measurable difference or jump, and the joy or sorrow which moved us so in the play of the actors on the *moving* film, becomes a static manifold of static pictures each differing measurably from its neighbour by a slightly more or less accentuated grimace. If we increase the number of pictures in a unit of ‘time’ by using a faster camera and then release this film at the ordinary speed, we get what is called slow motion pictures with which we are all familiar. In them we notice a much greater smoothness of movements which in life are jerky, as, for instance, the movements of a running horse. They appear smooth and non-jerky, the horse looks as if it were swimming. Indeed we do swim no less than fishes, except that our medium; namely, air, is less dense than water, and so our movements have to be more energetic to overcome gravitation. The above example is indeed the best analogy in existence of the working of our nervous system and of the difference between orders of abstractions. Let us imagine that some one wants to *study* some event as presented by the moving picture camera. What would he do ? He would first see the picture, in its moving, dynamic form, and later he would arrest the movement and devote himself to the contemplation of the static extensional manifold, or series, of the static pictures of the film. It should be noticed that the differences between the static pictures are finite, definite and *measurable.*

The power of analysis which we humans possess in our higher order abstractions is due precisely to the fact that they are *static* and so we can take our ‘time’ to investigate, analyse, . The lower order abstractions, such as our *looking* at the moving picture, are shifting and non-permanent and thus evade any serious analysis. On the level of *looking* at the *moving* film, we get a general *feeling* of the events, with a very imperfect memory of what we have seen, coloured to a large extent by our moods and other ‘emotional’, or organic states. We are on the shifting level of lower order abstractions, ‘feelings’, ‘motions’, and ‘emotions’. The first lower centres do the best they can in a given case but the value of their results is highly doubtful, as they are not especially reliable. Now the higher order abstractions are produced by the higher centres, further removed, and not in direct contact with the world around us. With the finite velocity of nerve currents it takes 'time' for impulses to reach these centres, as the cortical pathways offer higher neural resistances than the other pathways.[3] So there has to be a survival mechanism in the production of nervous means for arresting the stream of events and producing *static* pictures of permanent character, which may allow us to investigate, verify, analyse, . It must be noticed that because of this higher neural resistance of higher centres and the static character of the higher abstractions, these abstractions are less distorted by affective moods. For, since the higher abstractions persist, if we care to remember them, and the moods vary, we can contemplate the abstractions under different moods and so come to some *average* outlook on a given problem. It is true that we seldom do this, but we *may* do it, and this is of importance to us.

As one of the aims of the calculus is to study relative rates of change we will consider a series of successive values of our variable which differ by little from each other. If we have $y=f(x)$ we can consider the change in x for a short interval, let us say, from x_0 to x_1, so that we assign to our x two values, $x=x_0$ and $x=x_1$. The corresponding values of our function or y will be $y_0=f(x_0)$ and $y_1=f(x_1)$. In general, small changes in y will be almost proportional to the corresponding changes in x, provided $f(x)$ is 'continuous'.

Denoting the small increment of x by Δx, so that $x_1-x_0=\Delta x$ or $x_1=x_0+\Delta x$, function y receives the increment $y_1-y_0=\Delta y$ or $y_1=y_0+\Delta y$. Since $y_1=f(x_1)$ and $x_1=x_0+\Delta x$ we have:

$y_0+\Delta y = f(x_0+\Delta x)$; if we subtract

from both sides $y_0 = f(x_0)$ we would have

$\Delta y = f(x_0+\Delta x)-f(x_0)$; dividing both sides

by Δx we have
$$\frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \qquad (1)$$

The above ratio represents the ratio of the increment of the function to the increment of the variable. In the limit when the increment in the variable becomes vanishingly small or when Δx tends toward zero, and our function is continuous, the limit of this ratio gives us the law of change or growth of our function.

The limit which the ratio (1) approaches when Δx approaches 0,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \qquad (2)$$

is called the derivative of y with respect to x and is denoted by $D_x y$, which we read 'D_x of y', in symbols,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = D_x y \qquad (3)$$

Let us illustrate this by a simple numerical example. Take the equation $y=x^2$ and assume that $x=100$, whence $y=10{,}000$. Suppose the increment of x, namely, $\Delta x=1/10$. Then $x+\Delta x=100.1$ and $(x+\Delta x)^2 = 100.1\times 100.1=10020.01$. The last 1 is 1/100 and only one millionth part of the 10,000, and so, we can neglect it and consider $y+\Delta y=10{,}020$; whence $\Delta y=20$ and $\Delta y/\Delta x=20/0.1= 200$. In the general case, if $y=x^2$ and instead of x we take a slightly larger value, $x+\Delta x$, then our function y also becomes slightly larger; thus,

$$y+\Delta y =(x+\Delta x)^2=x^2+2x\Delta x+(\Delta x)^2$$

If we subtract $y=x^2$ from the last expression we have

$$\Delta y=2x\Delta x+(\Delta x)^2, \text{ dividing by } \Delta x, \text{ we have}$$

$$\Delta y/\Delta x = 2x+\Delta x.$$

In the limit as Δx approaches zero, the value of the above ratio, or the rate of change of our function, would be $2x$, as Δx would disappear. If our $x=100$, the above ratio would be 200, as determined above in the case of the numerical example. Another way of symbolizing the derivative is $D_x y=dy/dx$, but this requires a short explanation.

In Chapter XV we have already discussed the problem of the ‘infinitesimal’ and we have seen that ‘infinitesimal’ is a misnomer and that there is no such thing at all. Yet this word is very often uncritically used by mathematicians and is therefore often confusing. By an ‘infinitesimal’ mathematicians mean a *variable* which approaches zero as a limit. The condition that it should be a variable is essential. It would probably be better to call an ‘infinitesimal’ an *indefinitely* small quantity or ‘indefinitesimal’, and that is what the reader should understand when he sees anywhere the word ‘infinitesimal’ or ‘infinitely small quantity’.

These indefinitely small quantities are in general neither equal, nor even of one order. Some by comparison are indefinitely smaller than others, and hence are said to be ‘of higher order’. Usually several quantities are considered which approach zero simultaneously. In such a case one of them is chosen as the principal indefinitely small quantity. Let us recall that if we take any number, for example, 1, and divide it by 2 we have 1/2. If we divide 1 by 4 we have 1/4 which is smaller than 1/2; if we divide 1 by 10 we have 1/10 which is still smaller. If we carry this process on indefinitely, taking larger and larger denominators, the results are fractions of smaller and smaller values. In the limit, as the value of the denominator becomes indefinitely large the value of the fraction approaches zero. This simple consideration will help us in the classification of indefinitely small quantities.

Let us take a as the principal indefinitely small quantity and b another indefinitely small quantity. If the ratio b/a approaches zero with a we say that b is an indefinitely small quantity of higher order with respect to a. In other words, although a approaches zero in the limit yet it is infinitely larger than b and so the ratio b/a also approaches zero.

If the ratio b/a approaches a limit k different from zero as a approaches zero, then b is said to be of the ‘same order’ as a and $b/a=k+\varepsilon$ where ε is indefinitely small with respect to a. In such a case $b=a(k+\varepsilon)=ka+a\varepsilon$, and ka is called the principal part of b. The term $a\varepsilon$ is obviously of a higher order than a.

We may say in general that if we have a power of a, for instance a^n, such that the ratio b/a^n approaches a limit different from zero, b is called an ‘infinitesimal’ (indefinitesimal) of order n with respect to a.

Let us give a numerical illustration. We know that there are 60 minutes in one hour, 24 hours in a day, or that there are 1440 minutes in a day, and by multiplying 1440 by 7, that there are 10,080 minutes in a week. Our forefathers called this 1/10,080 part of a week a 'minute' because of its minuteness. It is obvious that a minute is very small as compared with a week. But if we subdivide a minute into 60 equal parts we have a still smaller quantity, a quantity of second order smallness and so we called it a second. Indeed there are 3600 seconds in one hour, 86,400 seconds in a day and 604,800 seconds in a week. If we decide that for some purpose a minute is as short a period of 'time' as we need to consider, then the second, 1/60 of a minute, is relatively so small that it could be neglected. In a calculation where 1/100 of some unit is the smallest value which needs to be considered, we may define this 1/100 as of first order

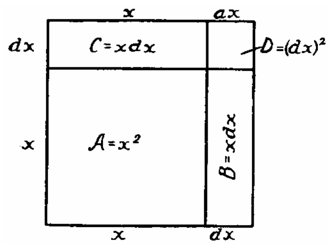


FIG. 1-A

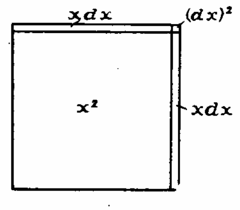


FIG. 1-B

smallness. Then 1/100 of 1/100, or 1/10,000, of that unit, which is relatively of second order smallness, is entirely negligible. The fractions whose smallness we are considering here are comparatively large, and we usually deal with much smaller quantities, but the smaller a quantity is, the more negligible the correspondingly smaller quantity of higher order becomes.

Let us consider a geometrical interpretation of the above. If we represent a quantity x by a line segment, and a slightly greater quantity, $x+dx$, by a slightly longer line segment; then the quantities x^2 and $(x+dx)^2=x^2+2xdx+(dx)^2$ may be represented by squares where sides are the line segments which represent the quantities x and $x+dx$ respectively.

If we denote the areas by A, B, C, D, we see that $A=x^2$ and that $A+B+C+D=x^2+2xdx+(dx)^2$. If we select our dx smaller and smaller the areas $B=C=xdx$ diminishing in one dimension only, become also smaller and smaller, but $D=(dx)^2$ is vanishing much more rapidly as it is diminishing in each of two dimensions, whence it is said to be a quantity of second order smallness, which for all purposes at hand may be neglected.

If we take $y=f(x)$ and its derivative

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = D_x y.$$

Then $\frac{\Delta y}{\Delta x} = D_x y + \varepsilon,$ where ε is an indefinitesimal,

and $\Delta y = D_x y \Delta x + \varepsilon \Delta x.$

In the above expression $D_x y\Delta x$ represents the principal part and $\varepsilon\Delta x$ appears as an indefinitesimal of higher order. This principal part is called the differential of y and is denoted by dy. If we choose $f(x)=x$ we have $dx=\Delta x$ and so, $dy = D_x y\, dx$.

So we see that the differential of the independent variable x is equal to the increment of that variable. This statement is not generally true about the dependent variable, as ε does not generally vanish.

The derivative is also sometimes denoted as $f'(x)$ or y' and this notation is due to Lagrange; all three notations are used and it is well to be acquainted with them.

The derivative of a function f(x) is in general another function of x, let us say $f'(x)$. If $f'(x)$ has a derivative, the new function is the derivative of the derivative-or the *second derivative* of $f(x)$ and is denoted by y'' or $f''(x)$. Similarly the third derivative y''' or $f'''(x)$ is defined as the derivative of the second derivative and so on. In the other notations we have:

$$D_x(D_x y)=D_x^2 y \text{ or } \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2 y}{dx^2}$$

Having introduced these few definitions it must be emphasized that the main importance of the calculus is in its central idea; namely, the study of a *continuous function* by following its history by *indefinitely small steps*, as the function *changes* when we give indefinitely small increments to the independent variable. As was emphasized before, the whole psycho-logics of this process is intimately connected with the activities of the *nervous structure* and also with the structure of science. In this work we are not interested in calculations, complications, or analytical niceties. Mathematicians have taken excellent care of all that. We need only to know about the structure and *method* which help to translate dynamic into static, and vice versa; to translate ‘continuity’ on one level, or order of abstraction, into ‘steps’ on another.

To illustrate what has been said and to give the reader the *feel* of the process, let us take for instance a simple equation $y=2x^3-x+5$ where y represents the function of the variable x expressed by a group of symbols to the right of the sign of equality.

To determine the relative rate of growth of this function, that is, to differentiate it, we replace x by a slightly larger value; namely, $x+\Delta x$, and see what happens to the expression. $2x^3$ becomes $2(x+\Delta x)^3=2x^3+6x^2\Delta x+6x(\Delta x)^2+2(\Delta x)^3$; $-x$ becomes $-x-\Delta x$ and the constant 5 remains unchanged. In symbols, $y+\Delta y=2x^3+6x^2\Delta x+6x(\Delta x)^2+2(\Delta x)^3-x-\Delta x+5$, where Δy represents the increment of the function and Δx represents the increment of the independent variable.

Subtracting the original expression $y=2x^3-x+5$ we get the amount by which the function has been increased, namely:

$$\Delta y = 6x^2\Delta x+6x(\Delta x)^2+2(\Delta x)^3-\Delta x.$$

To determine the *relation*, or *ratio*, of Δy, the increment of the function, to Δx the increment of the independent variable which produced Δy, we divide Δy by Δx, and obtain the equation

$$\frac{\Delta y}{\Delta x} = 6x^2 + 6x\Delta x + 2(\Delta x)^2 - 1.$$

Then as Δx approaches 0 the terms in the right-hand side of the equation which contain Δx as a factor also approach 0 and replacing the left-hand side by $\frac{dy}{dx}$ we obtain the equation $\frac{dy}{dx} = 6x^2 - 1$ which means, that as Δx approaches 0, the ratio of the increment of the function to the increment of the independent variable approaches $6x^2-1$, true for any value we may arbitrarily assign to x.

It should be noticed that in our function the left-hand side represents the 'whole' as composed of interrelated elements which are represented by the right-hand side. When instead of x we selected a slightly larger value; namely, $x+\Delta x$, we performed upon this altered value *all* the operations indicated by our expression. We thus have in mathematics, because of the self-imposed limitations, the first and only example of *complete* analysis, impossible in physical problems as in these there are always characteristics left out.

An important structural and methodological issue should also be emphasized. In the calculus we introduce a 'small increment' of the variable; we performed upon it certain indicated operations, and in the final results this arbitrary increment disappeared leaving important information as to the rate of change of our function. This device is structurally extremely useful and can be generalized and applied to language with similar results.

It has been noticed already that the calculus can be developed without any reference to graphs, co-ordinates or any appeal to geometrical notions; but as geometry is an all-important link between pure analysis and the outside world of physics, we find in geometry also the psycho-logical link between the higher and lower orders of abstraction. But the appeal to geometrical notions helps *intuition* and so is extremely useful. For this reason we will explain briefly a system of co-ordinates and show what geometrical significance the derivative has.

We take in a plane two straight lines $X'X$ and $Y'Y$, intersecting at O at right angles, so that $X'OX$ is horizontal extending to the left and right of O and YOY', is vertical, extending above and below O, as a frame of reference for the locations of point, lines, and other geometrical figures in the plane. We call this a two-dimensional rectangular system of co-ordinates. This method may be extended to three dimensions, and our points, lines, and other geometrical figures referred to a three-dimensional rectangular system of co-ordinates consisting of three mutually perpendicular and intersecting planes.

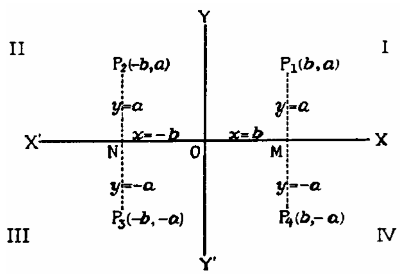


FIG. 2

As we see in Fig. 2, we have four quadrants I, II, III, IV, formed by the intersecting axes $X'X$ and $Y'Y$. The co-ordinates of a point P, by which we mean the distances from the axes determine the position of the point uniquely. We call *X'X* and *Y'Y* the *axis* of *X* and the *axis* of *Y* respectively, and *O* the *origin.* If we select a point P_1 in the plane of *X'X* and *Y'Y* and draw a line P_1M perpendicular to *X'X* then *OM* and MP_1 are called the co-ordinates of P_1; *OM* is called the *abscissa* and is denoted by $x=b$; and MP_1 *is* called the ordinate and denoted by $y=a$. We speak of P_1 as the point (b,a), *or,* in general, of any point as the point (x,y).

Let us draw $ON=OM=b$ and draw lines P_1P_4 and P_2P_3 through *M* and *N* respectively perpendicular to *X'X*, making $MP_4=NP_2=NP_3=MP_1=a$. We then have four points P_1, P_2, P_3, P_4, in each one of the four quadrants and all of them by construction would have equal numerical values for their abscissas and ordinates. To be able to discriminate between the four quadrants, and so avoid ambiguity, we make the convention that all values of *y* above *X'X* are to be positive and below *X'X* negative; and all values of *x* to the right of *Y'Y* positive, to the left negative. Thus we see that by such conventions the point P_1 would have both *b* and *a* positive; P_2 would have *b* negative and *a* positive; P_3 both *b* and *a* negative, and finally P_4 would have *b* positive and *a* negative, or in symbols $P_1(b,a)$; $P_2(-b,a)$; $P_3(-b,-a)$; and finally $P_4(b,-a)$.

It is obvious that for any point on the *X* axis (for instance *M)* the ordinate $y=0$. If our point is on the *Y* axis the abscissa $x=0$ and the co-ordinates of the origin *O* are both zero (0,0).

From the above definitions we see at once how to plot, or locate, a point. To plot the point (-4,3), since the abscissa *x* is negative and the ordinate *y* is positive we locate *N* on *X'X*, 4 units to the left of *O*. At *N* we erect a perpendicular upon which we locate the point (-4,3), 3 units above *N*. The symbol (-4,3) represents a particular case of the general symbol (x,y) and is accordingly plotted as a particular point as just shown. If instead of the pair or relations expressed by two equations $x=-4$, $y=3$, we have a single relation expressed by one equation, for example, $y=x-2$, we have *y* expressed as a function of *x*, whence by assigning to *x* different values, corresponding values of *y* are determined, and a set of points may be plotted where abscissas and ordinates are corresponding values of *x* and *y* respectively. Thus, when $x=0$, $y=-2$, when $x=1$, $y=-1$, when $x=2$, $y=0$, when $x=3$, $y=1$, when $x=4$, $y=2$, .

We may now plot the points *A*(0,-2); *B*(1,-1); *C*(2,0); *D*(3,1); *E*(4,2); or as many more points as we may choose by giving *x* additional different values.

If we give to *x* successive values with smaller differences our points would be closer together, for instance for

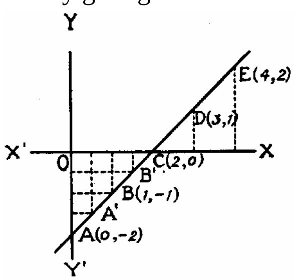


FIG. 3

$x=0$	$y=-2$	(*A*)
$x=0.5$	$y=-1.5$	(*A'*)
$x=1$	$y=-1$	(*B*)
$x=1.5$	$y=-0.5$	(*B'*)
$x=2$	$y=0$	(*C*)
. . .	. . .	

As we plot larger and larger numbers of points closer and closer together, in the limit, if we take indefinitely many such points, we approach a smooth line. It can be proved that an equation of the type given in this example; namely, where both variables are of the first order, always represents a *straight* line. Such equations are called therefore *linear* equations, as they represent straight lines.

The problem of linearity and non-linearity is of extreme importance, and we will return to it later on. Here we are interested only in the definition and meaning of linearity of equations.

Let us consider next a simple equation of second degree, $y=x^2/2$. In assigning arbitrary values to x, we note that x^2 is always positive (by the rule of signs) whether x is positive or negative. Hence, we may tabulate values of x with the double sign +/- meaning either + or -.

$x = 0$	$y=0$	(O)
$x = +/-1$	$y= ½$	(A)
$x = +/-2$	$y=2$	(B)
$x= +/-3$	$y=4½$	(C)
$x= +/-4$	$y=8$	(D)
. . .	. . .	

We see for each value of y we have two values for x which differ only in sign. This means that we have points on two sides of the Y axis with numerically equal abscissas and, since for x=0, y =0, the beginning of our curve is at the origin of co-ordinates and the curve is symmetrical with respect to the Y axis.

If we connect the points *D', C', B', A', O, A, B, C, D,* with straight lines we have a broken line. But if we choose smaller and smaller differences between the successive values of x, the broken line becomes smoother and smoother, and, in the limit, as we take increasingly smaller steps, or, in other words, plot indefinitely larger numbers of points in one interval, we approach a smooth, or continuous curve.

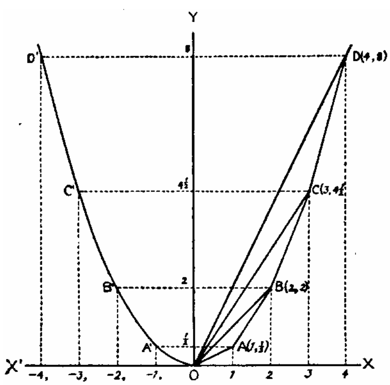


Fig. 4

It must be noticed that in equations of higher orders the ratio of changes in the function y to corresponding changes in the variable x vary from point to point, and so we have a *curve* instead of a straight line. It is necessary to become quite clear on this point so we may better compare the two different types of equations as to the law of their growth.

Let us write down in two columns the successive values for the two types of equations. Let us take the equation $y=\frac{x^2}{2}$ with the graph shown in the preceding diagram (Fig. 4) and the equation $y=2x$ as shown in Fig. 5.

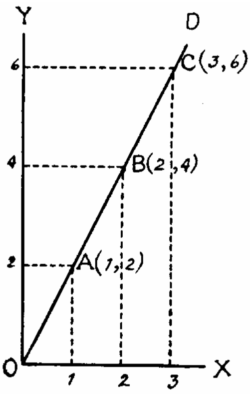


FIG. 5

Values of x	$y=2x$	$y=x^2/2$
-4	-8	8
-3	-6	4½
-2	-4	2
-1	-2	½
0	0	0
+1	+2	½
+2	+4	2
+3	+6	4½
+4	+8	8
+5	+10	12½
+6, .	+12, .	18, .

The equation $y=2x$ involves the variables in the first degree and we see that the ratio of changes in the ordinates to corresponding changes in the abscissas remains constant (proportional). The triangles in Fig. 5, are either equal or similar, which necessitates the equality of angles and so the line *OABCD* is of necessity a straight line. In this case as $x=0$ gave us $y=0$ the line passes through the origin of co-ordinates.

The picture is entirely different in the case of the higher degree equation, $y=\frac{x^2}{2}$, illustrated in Fig. 4. From the table of values of the function we see that the value of the function increases increasingly more rapidly than the values of the independent variable and so the ordinates are not *proportional* to the abscissas. If in Fig. 4 we connect *O* with *A, O* with *B, O* with C, O with *D,* respectively, we see that the lines *OA, OB, OC,* and *OD* have *different* angles with the axis *X'X*; the respective triangles are not similar, and so there is no proportionality. The lines *OA, OB, OC, OD.* , do *not* represent a straight line as they have all different angles with the axis *XX'* and so the points *A, B, C, D.* , cannot lie on a straight line but represent a *broken* line which, in the limit, when the points plotted become sufficiently near together, becomes a smooth and continuous curve.

The fact that equations in which the variables are only of the first degree, represent straight lines, and that equations of higher degrees represent curved lines is very important, as will appear later on. We must notice also that the problem of *linearity* is connected with *proportionality*.

These few simple notions concerning the use of co-ordinates will allow us to explain the geometrical meaning of the derivative and the differential.

Consider P_1 and P_2, (Fig. 6) two points on the curve, $y=f(x)$, referred to the axes OX and OY. Drop perpendiculars P_1M_1 and P_2M_2 from P_1 and P_2 to OX. These are the ordinates $y_1=f(x_1)$ and $y_2=f(x_2)$ of the points P_1 and P_2, and OM_1 and OM_2 are the abscissas x_1 and x_2 of the points P_1 and P_2. Through P_1 draw the secant P_1P_2, the tangent to the curve P_IT, and the line P_1Q parallel to OX. Then P_1Q represents $\Delta x=x_2-x_1$ the change in the variable x, and P_2Q represents $\Delta y=y_2-y_1=f(x_2)-f(x_1)$ the change in the function y.

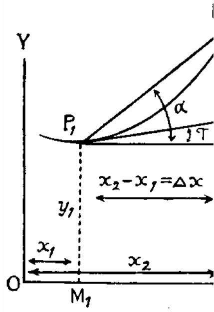


FIG. 6

In the right triangle P_1QP_2 the ratio P_2Q/P_1Q is a measure (the tangent) of the angle P_2P_1Q $(=\alpha)$ that is, $\tan\alpha=P_2Q/P_1Q=\Delta y/\Delta x=\dfrac{f(x_2)-f(x_1)}{\Delta x}$ or, since $x_2=x_1+\Delta x$

we may write $$\tan\alpha=\frac{f(x_2+\Delta x)-f(x_1)}{\Delta x}$$

As P_2 approaches P_1 along the curve, the secant P_1P_2 rotates about P_1 approaching P_1T as its limit, and the tangent of α approaches the tangent of τ, τ being the angle which P_1T, the tangent to the curve at P_1, makes with P_1Q. But as P_2 approaches P_1, $\Delta x=x_2-x_1=M_1M_2$ approaches zero or symbolically as $\Delta x\to 0$; $(\Delta y/\Delta x)\to\tan\tau$, that is $\tan\tau=\lim\limits_{\Delta x\to 0}(\Delta y/\Delta x)$. We see that the $\lim\limits_{x_2\to x_1}\dfrac{y_2-y_1}{x_2-x_1}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ represents nothing more or less than the derivative of the function representing the curve. In other words, the geometrical interpretation of the analytical process of differentiation is the finding of the slope of the graph of the function. The increment Δy of the function is represented by P_2Q; the differential dy is equal to NQ and $\Delta x = dx = P_1Q$; $\tan\angle TP_1Q=\dfrac{dy}{dx}$.

From the above considerations we see that the differential calculus gives, by the application of some extremely simple structural principles, a method of analysis by which we can discover a tendency at a particular stage rather than the final outcome after a definite interval. From such fundamental yet simple beginnings the whole calculus is developed. Most of these developments are not needed for our purpose, but we will explain one specially important theorem. The theorem in question is that the derivative of the sum of two functions is equal to the sum of their derivatives. In symbols

$$D_x(u+v)=D_xu+D_xv.$$

Let us symbolize $u+v=y$ and select a special value

$$y_0 = u_0 + v_0 \qquad (4)$$

then $y0+\Delta y = u_0 + \Delta u + v_0 + \Delta v$. By subtracting (4),

we have $\Delta y = \Delta u + \Delta v$. Dividing by Δx,

we have $\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$. When Δx approaches zero the left-hand side approaches $D_x y = D_x(u+v)$; and the first term of the right-hand approaches $D_x u$, while the second term approaches $D_x v$ and so,

$$D_x(u+v) = D_x u + D_x v.$$

The symbol D_x means also that certain operations are to be performed upon our function; namely, to find its derivative. When used in this sense it is called an operator. The operator D_x can be also written in its differential form as d/dx, and similarly for higher derivatives.

2. *MAXIMA AND MINIMA*

It will be useful to have some applications of the differential calculus explained.

If a function $y=f(x)$ is continuous in an interval $a<x<b$ and has larger (or smaller) values at some intermediate points than it has at or near the ends, then it has a maximum (or minimum) at some point $x=x_0$, inside this interval. If Fig. 7 represents the graph of the function, it is obvious that at the maximum (or minimum) the tangent to the curve is parallel to the axis and therefore the slope of this tangent is zero. As this slope is given by the derivative and the slope is zero we have a simple method of finding the maximum (or the minimum) of a function by equating the first derivative to zero; namely, $D_x y=0$ when $x=x_0$.

It is useful to be able to discriminate between the maximum and the minimum of a function. Fig. 7 shows that this can be done by finding means to discriminate between the two cases when our curve is concave upwards or concave downwards. The slope of a curve for a particular value of x is given by the value of $D_x y$, corresponding to that value of x. If the value $D_x y$ is positive, y increases as x increases, and the curve slopes up as we move to the right; if the value of $D_x y$ is negative, y decreases as x increases, and the curve slopes down as we move to the right.

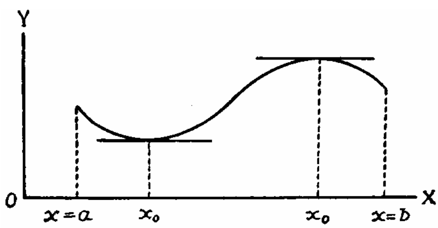


FIG. 7

If we consider the curve $y=f(x)$ which has its concave side turned upward (Fig. 8), the slope of the curve itself is a function of x, $\tan \alpha = f'(x)$. If we consider a variable point P on a curve $y=f(x)$, together with the tangent to the curve P, as following the curve in the direction of increasing values of x, the curve is concave upward whenever the slope is increasing algebraically, that is when $D_x \tan \alpha=0$. In other words, the curve is concave upwards for those values of x for which $D_x \tan \alpha$ is positive, or since $\tan \alpha = D_x y$ for those values of x for which $D_x \tan \alpha = D_x(D_x y) = D_x^2 y$ is positive. Similarly a curve is concave downwards for those values of x for which $D_x \tan \alpha = D_x(D_x y) = D_x^2 y$ is negative. These results can be expressed thus:

A curve $y=f(x)$ is concave upward when $D_x^2 y>0$, or, in words, when the second derivative is positive, and the curve is concave downward when the second derivative is negative, or, in symbols, when $D_x^2 y<0$.

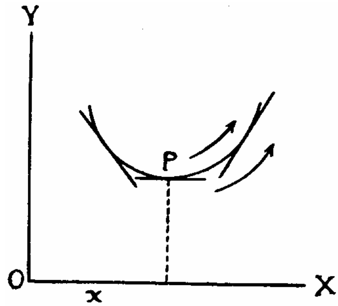


FIG. 8

From Fig. 7, we see that for a maximum x we must have our tangent parallel to the $X'X$ axis and our curve concave downwards, hence for these conditions the first derivative $[D_x y]_{x=x_0} = 0$, and the second derivative $[D_x^2 y]_{x=x_0} < 0$. For a minimum the first derivative must again be zero and the second derivative positive, whence the concave side of the curve is turned upwards. It should be noticed that the problems of maxima and minima play an extremely important structural psycho-logical and semantic role in our lives. All theories, somehow, are built on some minimum or maximum principle involving evaluations which are fundamental factors of all semantic reactions. In daily life we apply these structural and semantic notions continually. In science this tendency made its appearance quite early. The problem of maxima and minima was treated seriously as far back as the second century B.C. In the eighteenth century Maupertuis formulated a ‘supreme law of nature’, that in all natural processes the ‘action’ (energy multiplied by ‘time’) must be a minimum. Euler and Lagrange gave an exact basis and form to this principle; and finally Hamilton, in 1834, established this principle structurally as a *variational* principle, known as the hamiltonian principle, which appears to be of extreme generality and usefulness. It facilitates the derivation of the fundamental equations of mechanics, electrodynamics and electron theory. It has also survived, in a generalized form, the einsteinian revolution, for it contains nothing whatever which would connect it with a definite co-ordinate system; it involves only pure numbers and so is invariant to all transformations. It is structurally one of the most important invariants ascribed to nature, being independent of the systems of reference of the observers.

It is very desirable that this problem should be investigated further from the structural psycho-logical semantic and neurological point of view, as the very foundations of human psycho-logics are fundamentally connected with such a principle, which itself is an *invariant* in human psycho-logics.

Its importance is still increasing, and the hamiltonian principle plays a most remarkable role in all the newest advances of science. Any reader need only look attentively at his daily life to realize that there too this principle plays a predominant role.

3. *CURVATURE*

In modern scientific literature we hear often the fundamental term 'curvature' mentioned, and a few words about it will not be amiss. If we take two perpendicular lines *X'OX* and *OY* and select on *OY* a number of points *A, B, C, D.* , further and further away from *O* and describe arcs of circles with these points as centres with radii *AO. BO, CO, DO.* , respectively Fig. 9, we find each successive arc flatter and closer to the line *X'X* than its predecessor. In other words, the larger the radius of our circle, the flatter its arc is. In the limit as the radius of the circle becomes indefinitely large, the arc approaches a straight line by intuition and by definition. We notice also that the curvature of each circle is uniform, that is, one-valued at every point; but that when we pass from one circle to another of different radius, the curvature changes.

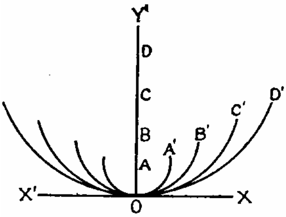


FIG. 9

If we consider a curve and two points on it, M_1 and M_2, (Fig. 10) and draw two tangents at these points; then the angle between these two tangents will depend on two factors, the sharpness of the curve and the distance between the points M_1 and M_2. If we take the points near enough and designate the length of the arc between them by Δs, the angle between the two tangents by $\Delta\theta$, then the limiting value of the ratio $\Delta\theta/\Delta s$, as M_2 approaches M_1, becomes $d\theta/ds$, and is a measure of the rate of change of the direction of the tangent at *M,* as *M* moves along the curve. Let us designate the rate at which the tangent turns where the point describes the curve with unit velocity as the curvature, or $k=\pm d\theta/ds$, but as *k* is essentially a positive number or zero we accept only the absolute value of this ratio. To find $d\theta/ds$ we notice that $\tan\theta=dy/dx$

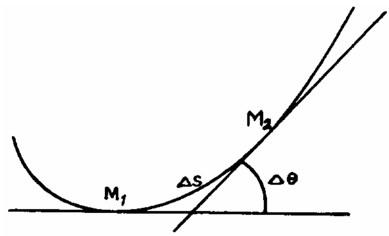


FIG. 10

or $\theta = \tan^{-1}\dfrac{dy}{dx} = \tan^{-1}y'$, whence $d\theta = \dfrac{dy'}{1+y'^2} = \dfrac{y''\,dx}{1+y'^2}$.

But $k = \dfrac{d\theta}{ds}$ where $ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+y'^2}\,dx$

whence $k = \dfrac{y''}{(1+y'^2)^{3/2}}$.

The reciprocal of the curvature is called the radius of curvature. The radius of curvature of a circle is its radius. The curvature of a curve is measured by the radius of the osculating circle, that circle which fits the curve the most closely in the neighbourhood of our point.

4. *VELOCITY*

Until now we have treated our independent variables as *any* quantity: but there are many problems where the independent variable represents ‘time’. For instance, if we travel by railroad the distance increases as ‘time’ increases, plants and animals grow with ‘time’, . By the average velocity with which a given point moves for a given length of ‘time’ we mean the distance *s* traversed divided by the ‘time’ elapsed. If, for instance, a train makes 5 miles in 10 minutes we say that its average velocity is 30 miles per hour, or, in symbols: Velocity, $v=\frac{s}{t}$. In this case we were considering uniform velocity, but very often we have to deal with velocities which are not uniform and which might be increasing or decreasing. In such a case we can describe the velocity approximately at any given moment if we take a short interval of ‘time’ immediately after the moment in question and take the average velocity for this short interval.

For instance, the distance a stone falls is according to the law, $s = 16\ t^2$. We want to find how fast it is going after t_1 seconds when $s_1 = 16\ t_1^2$, and a short interval after we have, let us say, $s_2 = 16\ t_2^2$. Obviously the average velocity for the interval t_2-t_1 is $\frac{s_2 - s_1}{t_2 - t_1}$ feet per second. If we take t_1=1, s_1=16, and the difference t_2-t_1=0.1 of a second then $s2=16t_2^2=16\times1.21=19.36$ and $\frac{s_2 - s_1}{t_2 - t_1} = \frac{19.36 - 16}{0.1} = \frac{3.36}{0.1}$ =33.6 ft. per second.

If we take the interval of ‘time’ smaller, for instance, 1/100 of a second we would have $\frac{s_2 - s_1}{t_2 - t_1}$ = 32.2 feet per second, and if we take the intervals as 1/1000 of a second the average velocity would be 32.0 feet per second. We see that we could determine the speed of the stone at any instant with any degree of accuracy by direct calculation, but this is not necessary. If we regard the interval t_2-t_1 as an increment of the variable *t*, that is as Δt, and $s_2 — s_1=\Delta s$ which represents the increment of the distance considered as a function of the ‘time’ we would have the average velocity $=\Delta s/\Delta t$. As Δt approaches zero in the limit, the average velocity approaches a limit and this limit is the velocity *v* at the instant t_1, or in symbols

$v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}$. In words, the velocity of a point is the ‘time’ derivative of the space traveled.

If the velocity is not uniform, the rate at which the velocity is increasing is called the acceleration and may be written as $a=dv/dt$, but as we have already seen *dv* is itself $d\left(\frac{ds}{dt}\right)$, hence $a=\frac{d^2s}{dt^2}$. In words, the acceleration is the second derivative of the distance with respect to ‘time’.

In the above notes we have not attempted to give the reader more than some structural and methodological notions, and what amount really to short structural explanations of definitions which will be useful later on. The reader can find many excellent books which give all the additional information he may want.

Section C. On the integral calculus.

So far, we have been studying a method by which to find the variation of a given function corresponding to an indefinitely small variation of our variable. We saw that the *rate of change* of our function was given by the first derivative, which in turn was also a function (usually different) of our independent variable and so could itself vary and have a rate of change, and so give us a second derivative, .

And now we must explain briefly the inverse problem; namely, given the derivative to find the function. In symbols, given $u=D_xU$, find U.

The function U is called the integral of u with respect to x, or, in symbols, $U = \int udx$.

To integrate a function $f(x)$ is to find a function $F(x)$ which when differentiated gives again the function $f(x)$ with which we started. As in this work we are not interested in computations, but only in the structural, methodological, and semantic aspects, the inverse problem of differentiation; namely, integration, is less important for us here, and I will explain only a single example. We have already differentiated the function $y=2x^3-x+5$ and found its derivative $dy/dx=6x^2-1$. Just as the derivative of the sum of a number of functions is equal to the sum of their derivatives, a similar rule holds for the integrals; namely, that the integral of the sum of a number of functions is equal to the sum of their integrals. Hence we can take in our example only the first term of our equation. In symbols $D_x(2x^3)=6x^2$; in words, the derivative of $2x^3$ is $6x^2$.

In a problem in integration we would have $6x^2$ given and we would have to find the original function from which $6x^2$ was obtained by differentiation. In our case the solution is already given; namely, $\int 6x^2dx = 2x^3$.* In general the solution of problems of integration is largely dependent on the ingenuity of the solver, although we have a number of standard formulae and methods. The geometrical meaning of integration is much more interesting for us and we will give a short explanation of it.

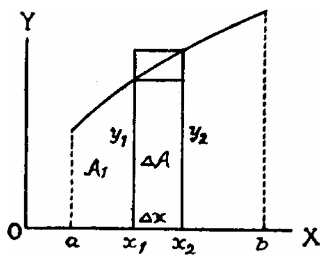


FIG. 11

If we consider the curve given by an equation $y=f(x)$ and the area bounded by the X axis, the two ordinates whose abscissas are $x=a$ and $x=b$ and the curve, we may find the area as follows:

* The constant of integration is omitted so as not to confuse the reader.

If we select an arbitrary value $x=x_1$ for which $y=y_1=f(x_1)$, denoting the corresponding value of the area A by A_1 (Fig. 11) and give to x_1 an increment Δx, then the area A_1 would receive the increment ΔA. We can approximate ΔA by the help of two rectangles, one of height $y_1=f(x_1)$, the other of height $y_2=y_1+\Delta y =f(x_2)=f(x_1+\Delta x)$.

We see that ΔA is larger than the smaller rectangle. In symbols

$$y_1\Delta x < \Delta A < (y_1 + \Delta y)\Delta x,$$

hence

$$y_1 < \frac{\Delta A}{\Delta x} < (y_1 + \Delta y).$$

As we pass to the limit and let Δx approach zero, we have

$\lim\limits_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = y_1$. That is, $D_xA=y_1=f(x_1)$ when $x=x_1$; which means that the ordinate of the curve at any point is equal to the x derivative of the area at that point. In general, $D_xA = y$, and hence, $A = \int y dx$.

The consideration of what is called the definite integral is still more instructive. Let us take the curve in Fig. 12 represented by an equation $y=f(x)$ and a pair of ordinates which intersect the X axis at the points $x=x_0$ and $x=x_n$.Let us divide the interval x_0x_n into n equal parts and erect ordinates at each point of division. Let us construct a set of pairs of rectangles with these ordinates as we constructed the single pair of rectangles in Fig. 11. By inspection of the figure we see that the area under the curve is slightly greater than the sum of the areas of the included rectangles and slightly less than the sum of the areas of the including rectangles. When n is allowed to increase without limit the sum of the areas of either set of these rectangles approaches the area bounded by the curve, the X axis, and the end ordinates. In symbols, the area of the first rectangle

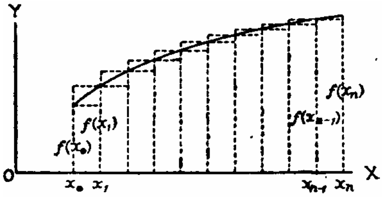


FIG. 12

beneath the curve is $f(x_0)\Delta x$, where Δx denotes $x_1 - x_0 = \frac{x_n - x_0}{n}$. The area of the second rectangle is $f(x_1)\Delta x$, . The sum of these areas is $f(x_0)\Delta x+f(x_1)\Delta x+ \ldots +f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x)\Delta x$.

If we allow n to increase without limit we have the area under the curve:

$$A = \lim_{n\to\infty} [f(x_0)\Delta x+f(x_1)\Delta x+ \ldots +f(x_{n-1})\Delta x] = \lim_{\Delta x\to\infty} \sum f(x)\Delta x$$

$$= \int_{x=x_0}^{x=x_n} f(x)dx = \Big[F(x)\Big]_{x=x_0}^{x=x_n} = F(x_n) - F(x_0)$$

In words, the above formula indicates the fundamental process of the integral calculus; namely: Let $f(x)$ be a continuous function of x throughout the interval $x_0 \leq x \leq x_n$. If we divide this interval into n equal parts by the points $x=x_0, x_1, \ldots, x_n$, and form the sum $f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$, as we let n increase without limit, this sum will approach a limit, which can be found by integrating the function $f(x)$, that is, by finding the function $F(x)$ of which $f(x)$ is the derivative, and by taking the integral between the limits $x=x_0$ and $x=x_n$; that is, by taking the difference between $F(x_n)$ and $F(x_0)$.

It must be noticed that in our first example, the case of the indefinite integral, we considered integration as the inverse of differentiation; in the second example, we considered the definite integral as the limit of a sum.

The symbol of the integral, $\int$, had its origin in the letter *S* from the latin word ‘summa’, the integral being historically understood as the definite integral, or the limit of a sum.

Section D. Further applications.

1. *PARTIAL DIFFERENTIATION*

When we have more than one independent variable, for example, two, we have to become acquainted with what is called partial differentiation. This process is important, as in practice we usually deal with several independent variables. It presents very little that is new from a structural and methodological point of view, but we give it here, simply to explain the meaning of the term, as the reader may find it used in other works.

If we have a function z of two independent variables x and y, $z=f(x,y)$ which geometrically represents a surface, we may differentiate with respect to one of the variables, let us say x, and hold the other variable y fast, that is, treat it as a constant. In this way we should then have a partial derivative of z with respect to x. Similarly, if we treat x as a constant and differentiate in respect to y, we should have the partial derivative of z with respect to y. The above definitions give us the rules for partial differentiation—that is, following the ordinary rules, considering each variable individually and treating all the other variables as constant.

The notation for partial derivatives is similar to the ones explained before, except that the lower case letter d is replaced by the script form ∂ or a subscript is used to indicate the variable with respect to which the differentiation is performed; for instance, $\frac{\partial f}{\partial x}=\frac{\partial z}{\partial x}=\partial z/\partial x=f'_x=z'_x=D_xf=D_xz,$. Higher derivatives are obtained without difficulty in like manner. If $z=f(x,y)$ and $\frac{\partial z}{\partial x}=f_x'(x,y)$ and $\frac{\partial z}{\partial y}=f_y'(x,y)$ the partial derivatives themselves are in general also functions of x and y and can in turn be differentiated. Thus, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}''(x,y)$, or $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}''(x,y)$. The order in which we differentiate is immaterial provided that the derivatives concerned are continuous. The *total differential* of a function of two variables, for example, $f(x,y)$, $df=d_xf+d_yf=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$ is equal to the sum of the partial differentials of the first order if we neglect terms of higher orders, whose values are indefinitely small quantities relative to the first order differentials. In symbols, $df = d_x f + d_y f = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy$. In words, the total differential of $f(x,y)$ is found by finding the partial derivatives with respect to x and y, multiplying them respectively by dx and dy, and adding.

2. *DIFFERENTIAL EQUATIONS*

A natural development of the invention of the calculus was the introduction of differential equations. Differential equations differ from the ordinary equations of mathematics in that in addition to variables and constants they contain also derivatives of one or more of the variables involved. Differential equations are of extreme importance, and arise in many problems. Newton solved his first differential equation in 1676 by the use of an infinite series, eleven years after his discovery of the calculus in 1665. Leibnitz solved his first differential equation in 1693, the year in which Newton first published his results. From this date on, progress in the development and application of differential equations was very rapid, and today the subject of differential equations occupies in the general field of mathematics a central position from which important and useful lines of development flow in many different directions.

To integrate or solve a differential equation means, analytically, to find all the functions which satisfy the equation. In geometry, it means to find all the curves which have the property expressed by the equation. In mechanics it means to find all the motions that may possibly result from a given set of forces, . The *degree* of the differential equation is defined as the degree of the derivative of the highest order which enters the equation. The *order* of the differential equations is the order of the highest derivative it contains.

Equations in x and y, of the first degree in y and its derivatives with respect to x, y', y''. , are called *linear equations.* The main equations of physics are *linear* differential equations of the second order, since y, the primitive function, y', the first derivative, and y'', the second derivative, appear only in the first degree. For instance the equation $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = X$, or $y''+a_1y'+a_2y = X$, when X represents a function of x alone is such an equation. It is linear, or of the first degree, because the second derivative, y'', appears only to the first degree. It is of the second order because that is the highest order derivative in the equation. As we may recall, the derivative of a function gives us the *rate of change* of the function when we give successive values to the independent variable. When we study the rate of change of the rate of change of our function, we study the rate of change of the first derivative which expresses the rate of change of the function, whence we obtain the derivative of the second order, and so on. If we equate our derivatives to zero, or choose a value of the variable for which our derivative becomes zero, the rate of change of our function becomes zero. In other words, the value of our function is momentarily constant, it has a stationary value.

Quite naturally, differential equations which involve derivatives involve implicitly and explicitly the whole fundamental structural framework of the calculus as explained in this chapter by expressing the ‘rate of change’ of some natural process. If the rate of change is zero, it might express some ‘natural law’, or some uniformity as found in nature. In other words, differential equations express differential laws, which in turn express the momentary tendencies of processes whose outcomes are given by the process of integration.

From what has already been said here, it is obvious that differential equations and the differential laws which they express are of extreme structural importance. They formulate not only the uniformities and tendencies found in nature, but also of necessity somehow involve causality. Besides which, they are also in accord with the physical structure and function of the nervous system. We shall return to this most important subject in the next chapter, in which we shall analyse the physical significance and aspects of what has been explained here.

3. *METHODS OF APPROXIMATION*

In discussing the above fundamental notions of the calculus we considered a portion *AB*, of the curve given by the equation $y=f(x)$, (Fig. 13) and two points on this curve P_1 with co-ordinates (x_1, y_1) and P_2 with co-ordinates (x_2, y_2) moving along the curve, the secant, or chord, P_1P_2 rotates about P_1, its length steadily diminishing, and in the limit as the length of the chord P_1P_2 tends toward zero, the slope of the secant approaches the slope of the tangent P_IT. We saw that the slope of this tangent was given by the value of the first derivative of the function which represented the curve. We were trying to get some knowledge of the direction of our curve at a given point by considering the slope of a *straight line* of smaller and smaller length. When we studied the curvature of our curve we considered the rate of change of the slope of our tangent and so, by the aid of a second derivative, we found the curvature. In this case we approximated our curve to a circle of radius equal to the radius of curvature of the curve at a given point.

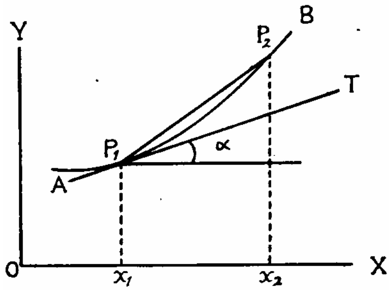


FIG. 13

In attempting to determine the length of a portion of our curve a point cannot be regarded as a piece of the curve but only as marking a position on it. For the purpose of determining the length of an arc it is convenient to replace each small element of the arc by its chord, a *lineal element.* By definition the length of an arc of a curve is the limit, if such limit exists, toward which the sum of the lengths of the chords of its small subdivisions tends as the number of chords increases indefinitely and their individual lengths all approach zero uniformly. For example, the circumference of a circle is the limit approached by the perimeter of an inscribed polygon as the number of its sides increases indefinitely, the lengths of the individual sides all approaching zero.

Similarly the length of a curve may be approximated by the sum of the lengths of segments of tangents at successive arbitrarily chosen points, merely by choosing the points nearer and nearer together. For example, the circumference of a circle is the limit approached by the perimeter of a circumscribed polygon as the number of its sides increases indefinitely, the lengths of the individual sides all approaching zero. In either case, a point on a curve taken with a vanishingly small portion of the tangent to the curve at that point may be called the *lineal element* of the curve.

The above definitions apply equally well in either two or three dimensions. The lineal element in two dimensions may be defined by three co-ordinates x, y, p, of which x and y are the co-ordinates of the point through which the lineal element passes and p is the slope of the element. This slope, as we already know, is to be found by differentiation, and is given by the formula $p=dy/dx$. In geometrical problems which relate the slope of a tangent to that of other lines, it is not the tangent that is of real importance but the *lineal element.* From this point of view a curve is made up of infinite numbers of vanishingly small lineal elements which are tangent to it, which is the point of view of the differential calculus. Or the curve is composed of infinite numbers of vanishingly small chords which are the sides of an inscribed polygon, which is the point of view of the integral calculus.

Obviously, in the limit, both points of view are equivalent, although as a matter of convenience they may be different. In any case, it must be obvious to the reader that using *straight lines* instead of pieces of a curve, or using as closer approximations arcs of circles, facilitates our study of the curves, indeed renders such study possible at all, and in practice we can carry our work to any degree of approximation we choose. But in theoretical work we require precision, hence we think in terms of infinite numbers of vanishingly small steps. The differential and integral calculus supply the only perfect technique for these processes of analysis and synthesis.

4. *PERIODIC FUNCTIONS AND WAVES*

We have already said that the most important relations of physics are represented by linear differential equations of the second order. It is important to know the connection of these equations with the general theory of waves or oscillations.

If on a circle of unit radius, as shown in Fig. 14, we take several points P_1, P_2, P_3, P_4, and connect these points by straight lines with the centre O, we get angles XOP_1, XOP_2, . In trigonometry we define certain functions of these angles and a unit of measurement. For our purpose we will only define the so-called sine and cosine, as we have already met the definition of tangent

$\tan\theta = \dfrac{M_1P_1}{OM_1}$. The angles XOP_1, XOP_2. , may be specified by the ratios M_1P_1/OP_1, M_2P_2/OP_2. , respectively each of which ratios has a definite value. This ratio in any case is called the *sine* of the angle, and is written in abbreviated form sin θ. If the radius of our circle is taken as unity then simply $M_1P_1 = \sin \angle P_1OM_1 = \sin\theta$, since $OP_1=1$. The ratio OM_1/OP_1 *is* called the *cosine* of the angle XOP_1, and is written cos θ.

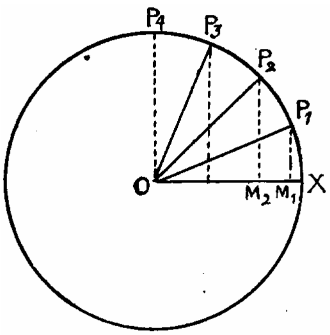


FIG. 14

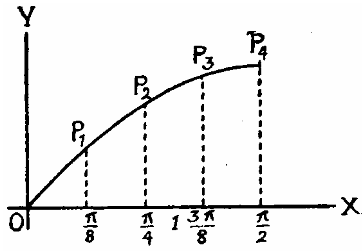


FIG. 15

There are two units of measurement of angles. In ordinary, or sexagesimal, measure, the unit angle is the degree, 1/360 of the entire angle about a point, 1/180 of a straight angle, or 1/90 of a right angle. The degree is divided into 60 equal parts called minutes. The minute is divided into 60 parts called seconds. In circular measure the unit angle is the radian, the angle at the centre of a circle whose arc is equal to the radius of the circle. This angle is a constant whether the circle be large or small, due to the fact that the circumferences of circles vary as their radii, and, in one circle, angles at the centre are proportional to their arcs. The constant ratio of the circumference of the circle to its radius is given by the number $\pi=3.14159\ldots$, this number being ‘incommensurable’ with unity. As the length of the circumference of a circle with radius *R*, is $2\pi R$ we see that the entire angle about the centre, which in degrees is 360, is in radians 2π; that a straight angle equals 180 degrees or π radians; and that a right angle equals 90 degrees or $\pi/2$ radians.

Thus 1 radian $= \dfrac{180°}{\pi} = 57°17'44''.806\ldots$ which, as it depends on the value of π, is itself an ‘irrational’ number. The ‘incommensurability’ of the radian with right and straight angles makes its practical use inconvenient. One of the main uses of the radian is in theory as it introduces a marked simplification in that the ratio of the sine of an indefinitely small angle to the angle itself is 1, when the angle is measured in radians. In other words; the equivalence of an indefinitely small arc and chord becomes apparent numerically when the angle and sine are expressed in one unit.

The following table gives the ordinary and radian measures, the sine, cosine and tangent of angles of 0, 1, 2, 3, and 4 right angles.

Angle in Right Angles	Angle in Degrees	Angle in Radians	Sine	Cosine	Tangent
0	0	0	0	1	0
1	90	$\pi/2$	1	0	$\pm\infty$
2	180	π	0	-1	0
3	270	$3\pi/2$	-1	0	$\pm\infty$
4	360	2π	0	1	0

From Fig. 14 and from this table, it follows that the values of the trigonometric functions are equal for the angles 0 and 2π, or in the language of degrees, for the angles 0° and 360°. We see also from Fig. 14 that the angle XOP_1, or any other angle, has one measure as expressed by its trigonometric functions if we add to it 360° or 2π radians.

The structural importance of the trigonometric functions in analysis lies in the fact that they are the *simplest* singly *periodic* functions and are therefore adapted for the representation of undulations. As we have already seen the sine and cosine have the single real period 2π, which means that they are not altered in value by the addition of 2π to the variable. The tangent has the period π.

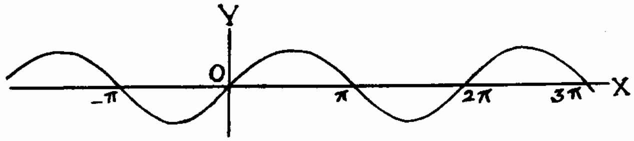
Besides the three functions defined above, we usually define three others, the secant, the cosecant and the cotangent as reciprocals respectively of the cosine, the sine, and the tangent. These last three we may disregard in our present discussion.

Let us consider the function $y=\sin x$, and construct the curve which this equation represents. If we draw a circle of *unit* radius, Fig. 14, the ordinates corresponding to the different angles XOP_1, XOP_2. , give the values of y, while the angles measured in radians, give the corresponding values of the abscissa x.

Plotting corresponding values of x and y as thus obtained in Fig. 14 we get in Fig. 15 the partial graph of the function $y = \sin x$. Proceeding again around our circle in Fig. 14, that is, adding 360° or 2π, to each of our angles, hence to their abscissas of the curve in Fig. 15, we add to the graph a second complete wave. We may thus proceed either forward or backward obtaining as many complete waves, or undulations, as we please, as in Fig. 16.

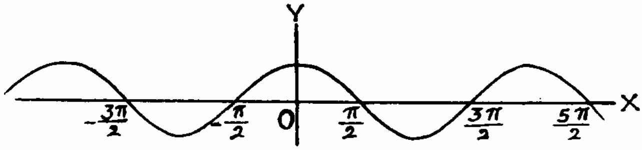
The curve represented by $y = \cos x$ is obtained in like manner and is quite similar to the sine curve. (See Fig. 17.)

To differentiate $\sin x$ we give to x the arbitrary values x_1, and $x_1+\Delta x$ and compute for y the corresponding values $y_1=\sin x_1$ and $y_1+\Delta y=\sin (x_1+\Delta x)$.



$y = \sin x$

FIG. 16



$y = \cos x$

FIG. 17

Subtracting y_1 from $y_1+\Delta y$, we have $\Delta y=\sin(x_1+\Delta x)-\sin x_1$. Dividing by Δx we have $$\Delta y/\Delta x = \frac{\sin(x_1+\Delta x)-\sin(x_1)}{\Delta x}.$$

To express the meaning of the above geometrically we may take a circle of unit radius and construct the angles x_1 and $(x_1+\Delta x)$, (Fig. 18). Then $M_1P_1=\sin x_1$, $M_2P_2=\sin(x_1+\Delta x)$, $QP_2=\sin(x_1+\Delta x)-\sin x_1=\Delta y$, and arc $P_1P_2=\angle P_1P_2=\Delta x$.

The limit approached by the ratio $\Delta y/\Delta x=QP_2/P_1P_2$ as P_2OP_1, or as $\Delta x\to 0$, is by previous definitions the sine of the angle $M_2P_2P_1$, since *in the limit* the arc P_1P_2 becomes a straight line, the hypotenuse of the right triangle P_1QP_2, which is similar to the right triangle P_1M_1O, whence angle $\angle M_2P_2P_1 = \angle x_1$. In other words, $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{QP_2}{P_1P_2} = \frac{OM_1}{OP_1}=\cos x$, or $D_x\sin x = \cos x$. It may be easily and similarly shown that $D_x\cos x = -\sin x$.

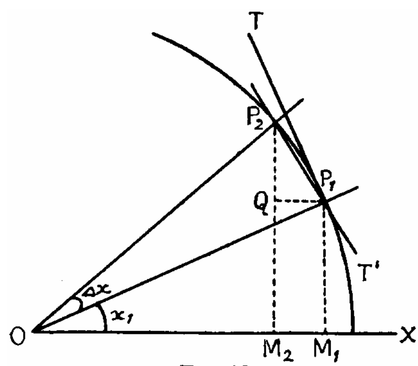


FIG. 18

Our main interest is in the second derivative. We see that the derivative of sin x is cos x, and that the derivative of cos x is -sin x.

Differentiating again we obtain the second derivative of sin x as -sin x, and the second derivative of cos x as -cos x.

The sine and cosine and their linear combinations are the only functions which when differentiated twice give us the second differential coefficient equal and of opposite sign to the original function. In symbols

$$\frac{d^2(\sin x)}{dx^2} = -\sin x;\ \text{and}\ \frac{d^2(\cos x)}{dx^2} = -\cos x.$$

In physics we deal with many processes which are structurally periodic, which means that a definite physical condition constantly recurs after equal intervals of ‘time’. The number of seconds or fractions of seconds within which the process runs its course is called the period. We know already that the simplest periodic functions are sine and cosine functions of the type

$\sin(x+2n\pi) = \sin x$, and

$\cos(x+2n\pi) = \cos x$, where n may have any integral value. Furthermore we have already seen that the first derivatives, therefore all derivatives of such functions are likewise simple sine and cosine functions. In particular, the second derivatives of the sine and cosine functions are likewise sine and cosine functions taken with the opposite algebraic sign.

If we express the variability of a process as a function of ‘time’, that is, by an equation of the form $S = F(t)$, then in a periodic process, $F(t_1+nT) = F(t_1)$, where T is the period and n any integer. If the process repeats itself, as in a periodic process, we must have

$$\left|\frac{dF}{dt}\right|_{t=t_1+nT} = \left|\frac{dF}{dt}\right|_{t=t_1} \text{ and } \left|\frac{d^2F}{dt^2}\right|_{t=t_1+nT} = \left|\frac{d^2F}{dt^2}\right|_{t=t_1} . ,$$

but, as we have already seen, the sine and cosine functions satisfy these conditions.

A process which can be described by an equation of the type $S = A \sin \frac{2\pi}{T}$ is called a *harmonic vibration* or, a ‘pure sine vibration’, or simply a ‘vibration’ or ‘oscillation’. The constant A, which represents the maximum value of the displacement on either side, is called the amplitude. The period T is called the ‘time of vibration’, its reciprocal value which gives the number of vibrations in a unit of ‘time’ is called the vibration number or *frequency*.

As the second derivatives of sine and cosine functions are equal to the original functions taken with the opposite signs, we can describe harmonic vibrations by differential equations of the first degree (linear) and of the second order of the special type $\frac{d^2S}{dt^2} = -a^2S$, where $S = A \sin(\frac{2\pi t}{T} + \varepsilon)$, A representing the amplitude, T the period, ε the phase of the vibration. The factor of proportionality a is taken as the square of any arbitrary real quantity to indicate that the right-hand side must always have the opposite sign to that of S.

The propagation of a vibration is called an advancing plane wave which has both velocity and direction.

Fourier has shown that any given form of wave may be represented by the superposition of a series of sine-waves, which gives sine-waves great theoretical and practical importance.

In writing this chapter I had two main aims. One was to briefly indicate the essential semantic factors involved in the differential methods. The other, to make the general reader and even specialists who are not mathematicians acquainted with some terms and rudiments of method which will be necessary for further discussion.

The differential methods involve semantic factors essential for a $\bar{A}$-system, the ∞-valued semantics of probability and for sanity and cannot be longer disregarded.

The main pressing issues are twofold. One, to formulate methods which would impart the $\bar{A}$ semantic reactions of the calculus, which need not involve any technicalities, and can be imparted in the most elementary home or school education. The other is to draw the attention of specialists to these semantic problems so that they will work them out.

An attempt to solve the first issue has been undertaken in the present volume. The second task will probably be accomplished in the not too distant future.

It is earnestly suggested to all scientists, professional men and teachers, who are not mathematicians, to become familiar with differential methods and so acquire the appropriate semantic reactions. Experience, in many cases, has shown that this will assist them in acquiring semantic balance and 'mental' efficiency. Teachers and physicians in particular, would be greatly helped in their efforts to train children and patients in the $\bar{A}$ reactions. The benefit is not in any 'calculations' whatsoever, but in the method and the related psycho-logical reactions.

There is an excellent, short, most elementary and amusing account of the calculus by Sylvanus P. Thompson *Calculus Made Easy* (Macmillan) which, for the present, is all that is needed for this purpose.