CHAPTER XIV

ON THE NOTION OF INFINITY

The questions on which there is disagreement are not trivialities; they are the very roots of the whole vast tree of modern mathematics. (22) E. T. BELL

The task of cleaning up mathematics and salvaging whatever can be saved from the wreckage of the past twenty years will probably be enough to occupy one generation. (22) E. T. BELL

The intention of the Hilbert proof theory is to atone by an act performed once for all for the continual titanic offences which Mathematics and all mathematicians have committed and will still commit against mind, against the principle of evidence; and this act consists of gaining the insight that mathematics, if it is not true, is at least consistent. Mathematics, as we saw, abounds in propositions that are not really significant judgments. (549)

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An objectivated property is usually called a *set* in mathematics. (549)

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If the objects are *indefinite* in number, that is to say if one is constantly exposed to seeing new and unforseen objects arise, it may happen that the appearance of a new object may require the classification to be modified and thus it is we are exposed to antinomies. *There is no actual* (*given complete*) *infinity*. (417) H. POINCARÉ

The structural notion of ‘infinite’, ‘infinity’, is of great semantic importance and lately has again become a subject of heated mathematical debates. My examination of this subject is from the point of view of a $\bar{A}$-system, general semantic, and a theory of sanity which completely eliminates identification. In Supplement III, I give a more detailed $\bar{A}$ analysis of the problem already anticipated by Brouwer, Weyl, Chwistek, and others. These problems are not yet solved, because mathematicians, in their orientations and arguments, still use *el*, *A* ‘logic’, ‘psychology’, and epistemology, which involve and depend on the ‘is’ of identity, making agreement impossible.

Mathematical infinity was first put on record by the Roman poet, Titus Lucretius, who, as far back as the first century B.C., wrote very beautifully about it in his *De Rerum Natura*.[1] As the author was a poet, and his work poetry, a few privileged literati had great pleasure in reading it; but this discovery, not being rigorously formulated, remained inoperative, and so practically worthless for mankind at large, for 2000 years. Only about fifty years ago, mathematical infinity was rediscovered by mathematicians, who formulated it rigorously, without poetry. Since then, mathematics has progressed with all other sciences in an unprecedented way. That this structural linguistic discovery was made so late is probably due to the usual blockage, the old *s.r*, old habits of 'thought', and prejudices.

In all arguments about infinity, from remote antiquity until Bolzano (1781-1848), Dedekind (1831-1916),and Cantor (1845-1918), there was a peculiar maxim involved. All arguments against infinity involved a certain structural assumption, which, at first inspection, seemed to be true and 'self-evident', and yet, if carried through, would be quite destructive to all mathematics existing at that date. Arguments favorable to infinity did not involve these tragic consequences. Quite naturally, mathematicians, and particularly Cantor, began to investigate this peculiar maxim and the *s.r* which were playing havoc. The structural assumption in question is that 'if a collection is part of another, the one which is a part must have fewer terms than the one of which it is a part'. This *s.r* was deeply rooted, and even found a scholarly formulation in Euclid's wording in one of his axioms: 'The whole is greater than any of its parts'. This axiom, although it is not an exact equivalent of the maxim stated above by loose reasoning, which was usual in the older days, could be said to imply the troublesome maxim. It is not difficult to see that the [E] axiom, as well as our troublesome maxim, expresses a structural generalization taken from experience which applies only to *finite* processes, arrays, . Indeed, both can be taken as a definition of finite processes, arrays, . It does *not* follow, however, that the one definition and structure must be true of infinite processes, arrays, . As a matter of fact, the break-down of this maxim gives us the precise definition of mathematical infinity. A process of generating arrays. , is called infinite when it contains, as parts, other processes, arrays. , which have 'as many' terms as the first process, array, .

The term 'infinite' means a process which does not end or stop, and it is usually symbolized by ∞. The term may be applied, also, to an array of terms or other entities, the production of which does not end or stop. Thus we may speak of the infinite process of generating number*s* because every positive integer, no matter how great, has a successor; we can also speak of infinite divisibility because the numerical technique gives us means to accomplish that. The term 'infinite' is used here as an adjective describing the characteristics of a *process*, but should never be used as a noun, as this leads to self-contradictions. The term 'infinity', as a noun, is used here only as an abbreviation for the phrase 'infinite process of generating number*s*', . If used in any other way than as an abbreviation for the full phrase, the term is meaningless in science (not in psychopathology) and should never be used. The above semantic restrictions are not arbitrary or purely etymological, but they follow the rejection of the ‘is’ of identity of a $\bar{A}$-system.

Before we can apply the term ‘infinite’ to physical processes, we must first theoretically elucidate this term to the utmost, and only then find out by experiment whether or not we can discover physical processes to which such a term can be applied. Fortunately, we have at our disposal a *semantic process* of generating number*s* which, by common experience, by definition, and by the numerical technique, is such that every number has a successor. Similarly, our semantic processes are capable by common experience, by definition, and by the numerical technique to divide a finite whole indefinitely. Thus, if we do not identify external physical objective processes with internal semantic processes, but differentiate between them and apply correct symbolism, we can see our way clear. If we *stop* this semantic process of generating number*s* at any stage, then we deal with a finite number, no matter how great; yet the process remains, by common experience, by definition, and by the numerical technique, such that it can proceed indefinitely. In the $\bar{A}$ sense, ‘infinite’, as applied to processes, means as much as ‘indefinite’. We should notice that the semantic *process* of *generating* number*s* should not be identified with a *selection* of *a* definite number, which, by necessity, is finite, no matter how great. The identification of the semantic process of generating number*s* with a definite number; the identification of the semantic process of infinite divisibility of finites in the direction of the small with the generating of number*s* in the direction of the great; and the identification of semantic internal processes with external physical processes. , are found at the foundation of the whole present mathematical scandal, which divides the mathematical world into two hostile camps.

The process of infinite divisibility is closely connected with the process of the infinite generation of number*s*. Thus we may have an array of *numbers* 1, 2, 3, . . . *n*, all of which are finite. The *semantic process* of *passing* from *n* to *n*+1 *is not* a number, but constitutes a characteristic of the semantic process. The *result* of the semantic process; namely, *n*+1, again becomes a finite number. If we take a fraction, *a*/*n*, the greater an *n* is selected, the smaller the fraction becomes, but with each selection the fraction again is finite, no matter how small.

Although the two processes are closely connected on the formal side, they are very different from the semantic point of view. The process of generating number*s* may be carried on indefinitely or ‘infinitely’ and has no upper limit, and we cannot assign such a limit without becoming tangled up in self-contradiction in terms. Not so with the process of indefinite or infinite divisibility. In this case, we start with a *finite*.

Existing mathematical symbolism and formalism lead to identification of both fundamentally different semantic processes and introduce a great deal of avoidable confusion. A $\overline{A}$ orientation will allow us to retain mathematical symbolism and formalism, but will not allow the identification of the semantic process of *passing* from number to number, which passing is not a number, with the *result* of this process which, in each case, becomes a *definite and finite number.*

It becomes obvious that the *A* terminology and present standard notions of 'number' identify the semantic *process* with its *result,* an identification which must ultimately be disastrous. The semantic process is thus potentially infinite, but the passing from n to $n + 1$ characterizes the semantic process, not number; number*s* representing only finite results of the indefinitely extended semantic process.

A $\overline{A}$ analysis without identification discloses, then, that only the semantic process can be indefinitely extended, but that the *results of* this process, or a number in *each case,* must be *finite.* To speak about an 'infinite' or, as it is called, 'transfinite' 'number', is to identify entirely different issues, and involves very definite self-contradictions in *m.o* terms. The existing mathematical terminology has been developed without the realization of $\overline{A}$ issues and the multiordinality of terms and leads automatically to such identifications. As long as mathematicians do not consider $\overline{A}$ issues, the problems of mathematical infinity will remain unsolved and hopeless; and yet, without a scientific theory of infinity, all of mathematics and most of science would be entirely impossible. A $\overline{A}$ clarification of these problems involves a new semantic definition of number*s* and mathematics, given in Chapter XVIII, which eliminates a great many mysteries in connection with mathematics and does not allow these dangerous and befogging identifications.

From a $\overline{A}$ point of view, we must treat infinity in the first cantorian sense; namely, as a *variable finite,* the term *variable* pertaining to the semantic process but *not* to number, the term *finite* pertaining to both the semantic *arrest* of the infinite semantic process, and so characterizing also its result; namely—a number.

In the meantime, the numerical technique is *indefinitely flexible* in the sense that no matter how great a number we take, we always can, by a semantic process, produce a greater number, and no matter how small the difference between two numbers might be, we always can find a third number which will be greater than the smaller, and smaller than the given greater number. Thus, we see that the *numerical technique* is such as to correspond in flexibility exactly to the *semantic processes,* but there is nothing flexible about a definite number once it is selected. What has been already said about a variable applies, also, to a number; namely, that a 'variable' does not 'vary' in the ordinary sense; but this term applies only to the semantic processes of the mathematician. The older intensional *A* definition of 'number' must have led to the older identifications. The $\overline{A}$, extensional, and *non-el* semantic definition of number*s* does not allow such identifications. The *A* term 'number' applied to a definite number, but also to an intensional definition of number*s*. The $\overline{A}$, or semantic definition of number*s*, is different in the sense that it finds extensional characteristics of each number, applicable to all number*s*, and so helps not to identify a definite number with the process of generating number*s*, which the use of one term for two entirely different entities must involve.

Cantorian *alephs,* then, are the result of identification or confusion of entirely different issues and must be completely eliminated. The rejection of *alephs* will require a fundamental revision of those branches of mathematics and physics which utilize them; yet, as far as I know, with a very few exceptions, the *alephs* are not utilized or needed, although the *'name'* is used, which spell-mark has become fashionable in many mathematical and physical circles. In the case of *alephs,* history may repeat itself and the *alephs,* like the 'infinitesimal', when their self-contradictory character becomes understood, will be eliminated without affecting the great body of mathematics, but only the small portions which are built on the *alephs.*

As to the existence of infinite processes, we know positively *only* about the *semantic process* of generating number*s* and the *semantic process* of infinite divisibility. These processes are evident in our common experience. We cannot *a priori* know if such infinite processes can be found in the world which must be discovered by investigation and experimentation.

The existing terminology is still *A* and is based on, and leads to, identification, and so in my $\overline{A}$ presentation I cannot use it and expect to clear up some of the issues involved. The terms such as 'class', 'aggregate', 'set'. , imply a definite static collection. The term 'infinite', in the meantime, can only be correctly and significantly used as applied to a dynamic semantic process. We cannot speak of 'infinite' classes, aggregates, sets. , and evade the issues of identification of entirely different entities. The term 'series' has a technical meaning in connection with numbers and so, for a general discussion of processes, is a little too specific. The term 'array' is more general, yet extensional, of which 'series' would be a special case. The general term 'number' is multiordinal and intensional and so, in the $\bar{A}$ extensional system, ∞-valued, and must be used in the plural; namely, 'number*s*'. The term 'number' in the singular will be used to indicate a definite number. The term 'denumerable' has been introduced by Cantor and means any extensional array of terms, facts, states, observables. , which can be put in one-to-one correspondence with the infinite array of positive integers.

Let me repeat once more: the semantic process may be carried on without limits, and the infinite series of positive integers is an extensional, technical, and verbal expression of this semantic process and the only infinite array of which existence we are certain.

We shall be able to explain, and to give a better definition of, mathematical infinity if we introduce an extremely useful structural term, 'equivalence'. Two processes, arrays. , between which it is possible to set up, by some law of transformation, a *one-to-one* correspondence are said to be *equivalent.* A process, array. , which is equivalent to a part of itself, is said to be infinite. In other words, a process, array, . which can be put into a *one-to-one* correspondence with a part of itself is said to be infinite. We can define a *finite* process, array. , (class, aggregate. ,) as one which is not infinite. The following is valid *exclusively* because of the use of the '**e t c .**'

A few examples will make this definition clearer. If we take the series of positive integers, 1, 2, 3, 4, . . . **e t c**., we can always double every number of this row *provided* we retain the *process-character, but not otherwise.* Let us write the corresponding row of their doubles under the row of positive integers, thus:

1, 2, 3, 4, 5, ... **etc.**
2, 4, 6, 8, 10, . . . **etc.**

Or we can treble them, or n-ble them, thus:

1, 2, 3, 4, 5, ... **etc.**
3, 6, 9, 12, 15, . . **etc.**

there are obviously as many numbers in each row below as in the row above, *provided we retain the* '***etc.***'*, so* the number*s* of number*s* in the two rows compared must be equal. All numbers which appear in each bottom row also occur in the corresponding upper row, although they only represent a *part* of the top row, *again provided that we retain the* '**etc**.'

The above examples show another characteristic of infinite processes, arrays, . In the first example, we have a *one-to-one* correspondence between the natural number*s* and the *even* number*s*, which are equal in number at each stage. Yet, the second row results from the first row by taking away all odd numbers, which, itself, represents infinite numbers of numbers.

This example was used by Leibnitz to prove that infinite arrays cannot exist, a conclusion which is not correct, since he did not realize that both finite and infinite arrays depend on definitions. We should be careful not to approach *infinite* processes, arrays. , with prejudices, or silent doctrines and assumptions, or, in general *s.r*, taken over from *finite* processes, arrays, .

Thus we see that the process of generating natural numbers is structurally an infinite process because its *results* can be put in a *one-to-one* correspondence with the results of the process of generating even numbers. , which is only a part of itself. Similarly, a line AB has infinitely many points, since its points can be put into a *one-to-one* correspondence with the points on a segment CD of AB. Another example can be given in the Tristram Shandy paradox of Russell. Tristram Shandy was writing his autobiography, and was using one year to write the history of one day. The question is, would Shandy ever complete his biography ? He would, provided he never died, or he lived infinite numbers of years. The hundredth day would be written in the hundredth year, the thousandth in the thousandth year, **etc**. No day of his life would remain unwritten, *again provided his process of living and writing would never stop.*

Such examples could be given endlessly. It is desirable to give one more example which throws some light on the problems of ‘probability’, ‘chance’, . The theory of probability originated through consideration of games of chance. Lately it has become an extremely important branch of mathematical knowledge, with fundamental structural application in physics, general semantics, and other branches of science. For instance, Boltzman based the second law of thermodynamics on considerations of probability. Boole’s ‘laws of thought’, and the many-valued ‘logic’ of Lukasiewicz and Tarski are also closely related to probability; and the new quantum mechanics uses it constantly, .

The term ‘probability’ may be defined in the rough as follows: If an event can happen in a different ways, and fails to happen in *b* different ways, and all these ways are equally likely to occur, the probability of the happening of the event is $\frac{a}{a+b}$, and the probability of its failing is $\frac{b}{a+b}$.

Let us assume that in a certain city a lecture is held each day, and that, though the listeners may change each day, the numbers of listeners are always equal. Suppose that one in each twenty inhabitants of this town has M as the first letter of his name. What is the probability that, *'by chance'*, all the names of the audience would begin with M ? Let us call such a happening the M-event. In the simplest case, when the daily number of listeners is only one, the probability of an M-event is 1 in 20, or 1/20. The probability of an M-event for an audience of 2 is 1 in $20 \times 20 = 400$, or 1/400. The probability that an audience of three members should have all three names begin with M would decrease twenty times further. Only once in 8000 lectures, on an average, would an M-event happen. For five people it would amount to 1 in $20\times20\times20\times20\times20=3{,}200{,}000$ days, or 1/3,200,000, or once in approximately 9000 *years;* for ten people, about once in thirty billion years; for twenty people, about once in a third of a quadrillion years. For one hundred people, the recurrence period of the M-event would be given as once in a number of years represented by more than a hundred figures. If the town, in this last example, should be as old as the solar system, and if the lectures had been delivered daily to an audience of one hundred people through this inconceivably long period, the probability is extremely small that the M-event would happen at all.[2]

From the human, *anthropomorphic,* point of view, we would say that such an event is impossible. But it must be remembered that this is only an anthropomorphic point of view, and our judgements are coloured by the temporal scale of our own lives. Of course, to carry such an anthropomorphic viewpoint into cosmic speculations is simply silly, a survival of the primitive structure of language and its progeny—metaphysics and mythologies.

The theory of infinity throws considerable structural light on such primitive speculations. In this external world, we deal with processes, and, as we measure 'length' by comparison with freely selected convenient units of 'length', let us say, an inch; or we measure 'volume' by freely selected convenient units of 'volume'; so, also, we compare *processes* with some freely selected and convenient *unit-process.* The diurnal rotation of our earth is such a process, and, if we choose, we can use it as a measuring unit or as a comparison standard. Of late, we have become aware that the rotation of the earth is not quite regular, and so, for accurate measurements, the old accepted unit-process of a day, or its subdivision, a second, is not entirely satisfactory For scientific purposes, we are trying to find some better unit-process, but we have difficulty, as the problem is naturally circular. When we speak in terms of a 'number of years', or of seconds, we speak about perfectly good observational experimental facts, about quite definite relations, the best we know in

1933. We do not make any metaphysical assertions about ‘time’ and we should not be surprised to find that statements involving ‘years’ are generally propositions, but that statements involving ‘time’ often are not. It is necessary not to forget this to appreciate fully what follows.

The theory of infinity will clear away a troublesome stumbling-block. We will use the expression ‘infinite numbers of years’, remembering the definition of ‘infinite numbers’ and what was said about the *unit-process* which we call a year. We have seen in an example above that if only a hundred individuals attend a lecture, and all ‘by chance’ have their names begin with M, such an event happens, on an average, only once in an inconceivably large number of years, represented by a number with a hundred figures. If we would ask *how many* times an occurrence would happen, we would have to state the period in years for which we ask the *how many*. It is easy to see that in infinite numbers of years, this humanly extremely rare occurrence would happen precisely *infinite numbers of times*, or, in other words, ‘just as often’, this last statement being from a non-anthropomorphic point of view. An event that appears, from our human, limited, anthropomorphic point of view, as ‘rare’, or as ‘chance’, when transposed from the level of finite process, arrays. , to that of infinite processes, arrays. , is as ‘regular’, as much a ‘law’, involving ‘order’, as anything else. It is the old primitive *s.r* to, suppose that man is the only measure of things.

Here the reader might say that infinite numbers of years is a rather large assumption to be accepted so easily. This objection is indeed serious, but a method which can dispose of it is given later on. At this stage, it is sufficient to say that, on the one hand, this problem is connected with the semantic disturbance, called identification (objectification of ‘time’), which afflicts the majority of us, excepting a few younger einsteinists; and that, on the other hand, it involves the structurally reformulated law of the ‘conservation of energy’, ‘entropy’, .

Before parting with the problem of infinity, let me say a word about: the notion of ‘continuity’, which is fundamental in mathematics. Mathematical continuity is a structural characteristic connected with ordered series. The difficulties originated in the fact that a ‘continuous’ series. must have infinite numbers of terms between any two terms. Accordingly, these difficulties are concerned with infinity. That mathematicians. need some kind of contiguity is evident from the example of two intersecting lines. If the lines have gaps, as, for instance, — — — — , there would be the possibility that two gaps would coincide, and the two lines not intersect; although in a plane the first line would pass to the other side of the second line. At present, we have two kinds of ‘continuity' used in mathematics. One is a supposedly 'high-grade' continuity; the other, supposedly, is a 'low-grade' continuity, which is called 'compactness' or 'density', with the eventual possibility of gaps. I am purposely using rather vague language, since these fundamental notions are now being revised, with the probability that we shall have to be satisfied with 'dense' or 'compact' series and abandon the older, perhaps delusional, 'high-grade' continuity. It is interesting to note that the differential and integral calculus is supposedly based on the 'high-grade' continuity, but the calculus will not be altered if we accept the 'low-grade' compactness, all of which is a question of an A or $\bar{A}$ orientation.

Vague feelings of 'infinity' have pervaded human *s.r* as far back as records go. Structurally, this is quite natural because the term infinity expresses primarily a most important semantic process. The majority of our statements can also be reformulated in a language which explicitly involves the term 'infinity'. An example has been already given when we were speaking about the universal propositions which were supposed to be of *permanent* validity, in other *language*, valid for 'infinite numbers of years'. We see how the trick is done—a vague quasi-qualitative expression like 'permanent' or 'universal' is translated into a quantitative language in terms of 'numbers of years'. Such translation of qualitative language into quantitative language is very useful, since it allows us to make more precise and definite the vague, primitive structural assumptions, which present enormous semantic difficulties. This brings to our attention more clearly the structural facts they supposedly state, and aids analysis and revision. In many instances, such translations make obvious the illegitimacy of the assumptions of 'infinite velocities' and so clear away befogging misunderstandings, and beneficially affect our *s.r*.